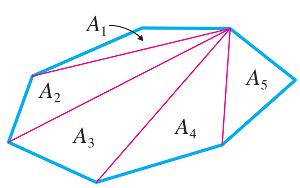
12 Limits: A Preview of Calculus

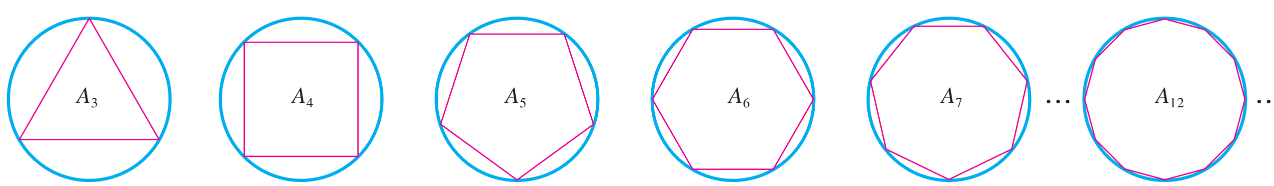
Chapter Overview



$A = A_1 + A_2 + A_3 + A_4 + A_5$

In this chapter we study the central idea underlying calculus—the concept of *limit*. Calculus is used in modeling numerous real-life phenomena, particularly situations that involve change or motion. To understand the basic idea of limits let's consider two fundamental examples.

To find the area of a polygonal figure we simply divide it into triangles and add the areas of the triangles, as in the figure to the left. However, it is much more difficult to find the area of a region with curved sides. One way is to approximate the area by inscribing polygons in the region. The figure illustrates how this is done for a circle.



If we let A_n be the area of the inscribed regular polygon with n sides, then we see that as n increases A_n gets closer and closer to the area of the circle. We say that the area A of the circle is the *limit* of the areas A_n and write

$$\text{area} = \lim_{n \to \infty} A_n$$

If we can find a pattern for the areas A_n, then we may be able to determine the limit A exactly. In this chapter we use a similar idea to find areas of regions bounded by graphs of functions.

In Chapter 2 we learned how to find the average rate of change of a function. For example, to find average speed we divide the total distance traveled by the total time. But how can we find *instantaneous* speed—that is, the speed at a given instant? We can't divide the total distance traveled by the total time, because in an instant the total distance traveled is zero and the total time spent traveling is zero! But we can find the *average* rate of change on smaller and smaller intervals, zooming in on the instant we want. For example, suppose $f(t)$ gives the distance a car has traveled at time t. To find the speed of the car at exactly 2:00 P.M., we first find the average speed on an interval from 2 to a little after 2, that is, on the interval $[2, 2 + h]$. We know that the average speed on this interval is $[f(2 + h) - f(2)]/h$. By finding this average speed

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for smaller and smaller values of h (letting h go to zero), we zoom in on the instant we want. We can write

$$\text{instantaneous speed} = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}$$

If we find a pattern for the average speed, we can evaluate this limit exactly.

The ideas in this chapter have wide-ranging applications. The concept of "instantaneous rate of change" applies to any varying quantity, not just speed. The concept of "area under the graph of a function" is a very versatile one. Indeed, numerous phenomena, seemingly unrelated to area, can be interpreted as area under the graph of a function. We explore some of these in *Focus on Modeling*, page 929.

12.1 Finding Limits Numerically and Graphically

In this section we use tables of values and graphs of functions to answer the question, What happens to the values $f(x)$ of a function f as the variable x approaches the number a?

Definition of Limit

We begin by investigating the behavior of the function f defined by

$$f(x) = x^2 - x + 2$$

for values of x near 2. The following table gives values of $f(x)$ for values of x close to 2 but not equal to 2.

x	$f(x)$
1.0	2.000000
1.5	2.750000
1.8	3.440000
1.9	3.710000
1.95	3.852500
1.99	3.970100
1.995	3.985025
1.999	3.997001

x	$f(x)$
3.0	8.000000
2.5	5.750000
2.2	4.640000
2.1	4.310000
2.05	4.152500
2.01	4.030100
2.005	4.015025
2.001	4.003001

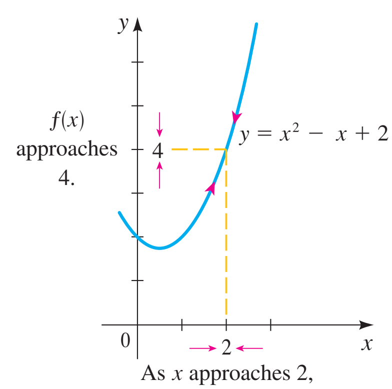


Figure 1

From the table and the graph of f (a parabola) shown in Figure 1 we see that when x is close to 2 (on either side of 2), $f(x)$ is close to 4. In fact, it appears that we can make the values of $f(x)$ as close as we like to 4 by taking x sufficiently close to 2. We express this by saying "the limit of the function $f(x) = x^2 - x + 2$ as x approaches 2 is equal to 4." The notation for this is

$$\lim_{x \to 2} (x^2 - x + 2) = 4$$

In general, we use the following notation.

Definition of the Limit of a Function

We write

$$\lim_{x \to a} f(x) = L$$

and say

"the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a, but not equal to a.

Roughly speaking, this says that the values of $f(x)$ get closer and closer to the number L as x gets closer and closer to the number a (from either side of a) but $x \neq a$.

An alternative notation for $\lim_{x \to a} f(x) = L$ is

$$f(x) \to L \qquad \text{as} \qquad x \to a$$

which is usually read "$f(x)$ approaches L as x approaches a." This is the notation we used in Section 3.6 when discussing asymptotes of rational functions.

Notice the phrase "but $x \neq a$" in the definition of limit. This means that in finding the limit of $f(x)$ as x approaches a, we never consider $x = a$. In fact, $f(x)$ need not even be defined when $x = a$. The only thing that matters is how f is defined *near* a.

Figure 2 shows the graphs of three functions. Note that in part (c), $f(a)$ is not defined and in part (b), $f(a) \neq L$. But in each case, regardless of what happens at a, $\lim_{x \to a} f(x) = L$.

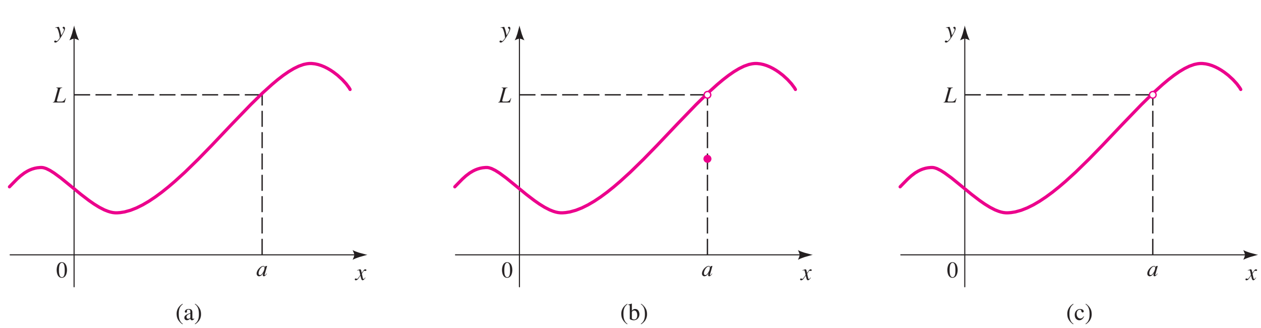


Figure 2
$\lim_{x \to a} f(x) = L$ in all three cases

Estimating Limits Numerically and Graphically

In Section 12.2 we will develop techniques for finding exact values of limits. For now, we use tables and graphs to estimate limits of functions.

Example 1 Estimating a Limit Numerically and Graphically

Guess the value of $\lim_{x\to 1} \dfrac{x-1}{x^2-1}$. Check your work with a graph.

Solution Notice that the function $f(x) = (x-1)/(x^2-1)$ is not defined when $x = 1$, but this doesn't matter because the definition of $\lim_{x\to a} f(x)$ says that we consider values of x that are close to a but not equal to a. The following tables give values of $f(x)$ (correct to six decimal places) for values of x that approach 1 (but are not equal to 1).

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

$x > 1$	$f(x)$
1.5	0.4000000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

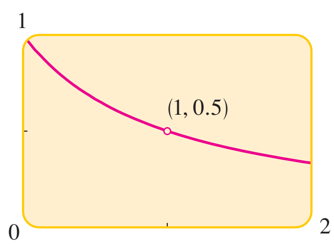


Figure 3

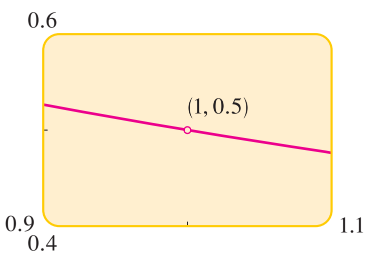


Figure 4

On the basis of the values in the two tables, we make the guess that

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = 0.5$$

As a graphical verification we use a graphing device to produce Figure 3. We see that when x is close to 1, y is close to 0.5. If we use the ZOOM and TRACE features to get a closer look, as in Figure 4, we notice that as x gets closer and closer to 1, y becomes closer and closer to 0.5. This reinforces our conclusion. ■

Example 2 Finding a Limit from a Table

Find $\lim_{t\to 0} \dfrac{\sqrt{t^2+9}-3}{t^2}$.

Solution The table in the margin lists values of the function for several values of t near 0. As t approaches 0, the values of the function seem to approach 0.1666666 . . . , and so we guess that

$$\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \frac{1}{6}$$ ■

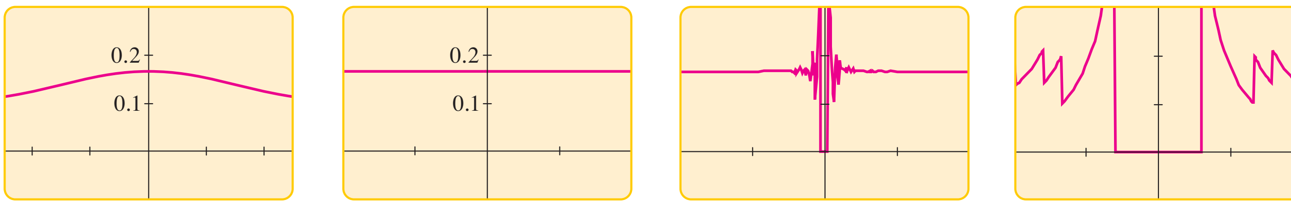
t	$\dfrac{\sqrt{t^2+9}-3}{t^2}$
± 1.0	0.16228
± 0.5	0.16553
± 0.1	0.16662
± 0.05	0.16666
± 0.01	0.16667

t	$\dfrac{\sqrt{t^2+9}-3}{t^2}$
± 0.0005	0.16800
± 0.0001	0.20000
± 0.00005	0.00000
± 0.00001	0.00000

What would have happened in Example 2 if we had taken even smaller values of t? The table in the margin shows the results from one calculator; you can see that something strange seems to be happening.

If you try these calculations on your own calculator, you might get different values, but eventually you will get the value 0 if you make t sufficiently small. Does this mean that the answer is really 0 instead of $\frac{1}{6}$? No, the value of the limit is $\frac{1}{6}$, as we will show in the next section. The problem is that the calculator gave false values because $\sqrt{t^2+9}$ is very close to 3 when t is small. (In fact, when t is sufficiently small, a calculator's value for $\sqrt{t^2+9}$ is 3.000 . . . to as many digits as the calculator is capable of carrying.)

Something similar happens when we try to graph the function of Example 2 on a graphing device. Parts (a) and (b) of Figure 5 show quite accurate graphs of this function, and when we use the TRACE feature, we can easily estimate that the limit is about $\frac{1}{6}$. But if we zoom in too far, as in parts (c) and (d), then we get inaccurate graphs, again because of problems with subtraction.



(a) $[-5, 5]$ by $[-0.1, 0.3]$ (b) $[-0.1, 0.1]$ by $[-0.1, 0.3]$ (c) $[-10^{-6}, 10^{-6}]$ by $[-0.1, 0.3]$ (d) $[-10^{-7}, 10^{-7}]$ by $[-0.1, 0.3]$

Figure 5

Limits That Fail to Exist

Functions do not necessarily approach a finite value at every point. In other words, it's possible for a limit not to exist. The next three examples illustrate ways in which this can happen.

Example 3 A Limit That Fails to Exist (A Function with a Jump)

The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

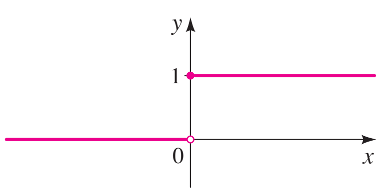


Figure 6

[This function is named after the electrical engineer Oliver Heaviside (1850–1925) and can be used to describe an electric current that is switched on at time $t = 0$.] Its graph is shown in Figure 6. Notice the "jump" in the graph at $x = 0$.

As t approaches 0 from the left, $H(t)$ approaches 0. As t approaches 0 from the right, $H(t)$ approaches 1. There is no single number that $H(t)$ approaches as t approaches 0. Therefore, $\lim_{t\to 0} H(t)$ does not exist. ■

Example 4 A Limit That Fails to Exist (A Function That Oscillates)

Find $\lim_{x\to 0} \sin \dfrac{\pi}{x}$.

Solution The function $f(x) = \sin(\pi/x)$ is undefined at 0. Evaluating the function for some small values of x, we get

$$f(1) = \sin \pi = 0 \qquad f\left(\tfrac{1}{2}\right) = \sin 2\pi = 0$$
$$f\left(\tfrac{1}{3}\right) = \sin 3\pi = 0 \qquad f\left(\tfrac{1}{4}\right) = \sin 4\pi = 0$$
$$f(0.1) = \sin 10\pi = 0 \qquad f(0.01) = \sin 100\pi = 0$$

Similarly, $f(0.001) = f(0.0001) = 0$. On the basis of this information we might be tempted to guess that

$$\lim_{x\to 0} \sin \frac{\pi}{x} \overset{?}{=} 0$$

but this time our guess is wrong. Note that although $f(1/n) = \sin n\pi = 0$ for any integer n, it is also true that $f(x) = 1$ for infinitely many values of x that approach 0. (See the graph in Figure 7.)

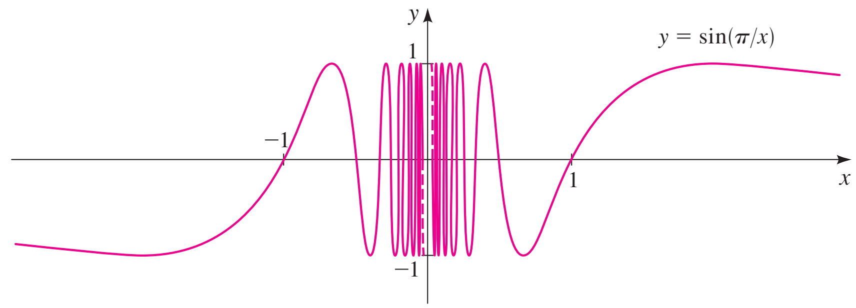


Figure 7

The broken lines indicate that the values of $\sin(\pi/x)$ oscillate between 1 and -1 infinitely often as x approaches 0. Since the values of $f(x)$ do not approach a fixed number as x approaches 0,

$$\lim_{x \to 0} \sin \frac{\pi}{x} \quad \text{does not exist}$$ ■

Example 4 illustrates some of the pitfalls in guessing the value of a limit. It is easy to guess the wrong value if we use inappropriate values of x, but it is difficult to know when to stop calculating values. And, as the discussion after Example 2 shows, sometimes calculators and computers give incorrect values. In the next two sections, however, we will develop foolproof methods for calculating limits.

Example 5 A Limit That Fails to Exist (A Function with a Vertical Asymptote)

Find $\lim_{x \to 0} \dfrac{1}{x^2}$ if it exists.

Solution As x becomes close to 0, x^2 also becomes close to 0, and $1/x^2$ becomes very large. (See the table in the margin.) In fact, it appears from the graph of the function $f(x) = 1/x^2$ shown in Figure 8 that the values of $f(x)$ can be made arbitrarily large by taking x close enough to 0. Thus, the values of $f(x)$ do not approach a number, so $\lim_{x \to 0}(1/x^2)$ does not exist.

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000

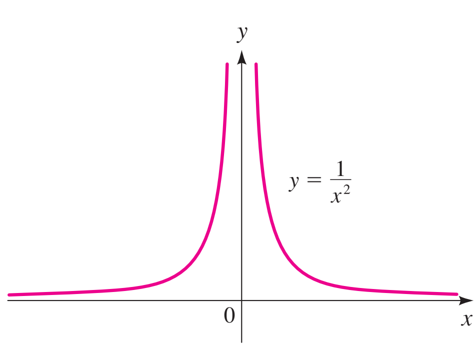


Figure 8 ■

To indicate the kind of behavior exhibited in Example 5, we use the notation

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

This does not mean that we are regarding ∞ as a number. Nor does it mean that the limit exists. It simply expresses the particular way in which the limit does not exist: $1/x^2$ can be made as large as we like by taking x close enough to 0. Notice that the line $x = 0$ (the y-axis) is a vertical asymptote in the sense we described in Section 3.6.

One-Sided Limits

We noticed in Example 3 that $H(t)$ approaches 0 as t approaches 0 from the left and $H(t)$ approaches 1 as t approaches 0 from the right. We indicate this situation symbolically by writing

$$\lim_{t \to 0^-} H(t) = 0 \qquad \text{and} \qquad \lim_{t \to 0^+} H(t) = 1$$

The symbol "$t \to 0^-$" indicates that we consider only values of t that are less than 0. Likewise, "$t \to 0^+$" indicates that we consider only values of t that are greater than 0.

Definition of a One-Sided Limit

We write

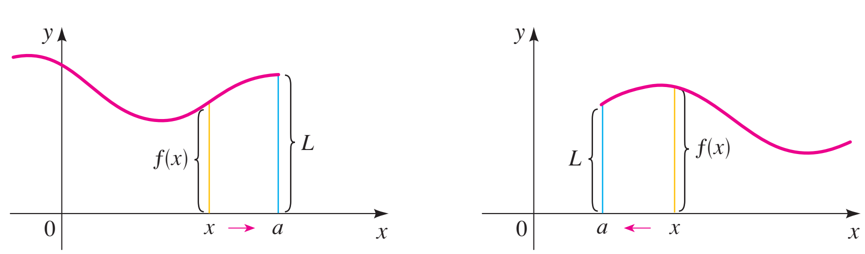
$$\lim_{x \to a^-} f(x) = L$$

and say the "left-hand limit of $f(x)$ as x approaches a" [or the "limit of $f(x)$ as x approaches a from the left"] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a.

Notice that this definition differs from the definition of a two-sided limit only in that we require x to be less than a. Similarly, if we require that x be greater than a, we get "the **right-hand limit of $f(x)$ as x approaches a** is equal to L" and we write

$$\lim_{x \to a^+} f(x) = L$$

Thus, the symbol "$x \to a^+$" means that we consider only $x > a$. These definitions are illustrated in Figure 9.



(a) $\lim_{x \to a^-} f(x) = L$

(b) $\lim_{x \to a^+} f(x) = L$

Figure 9

By comparing the definitions of two-sided and one-sided limits, we see that the following is true.

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L$$

Thus, if the left-hand and right-hand limits are different, the (two-sided) limit does not exist. We use this fact in the next two examples.

Example 6 Limits from a Graph

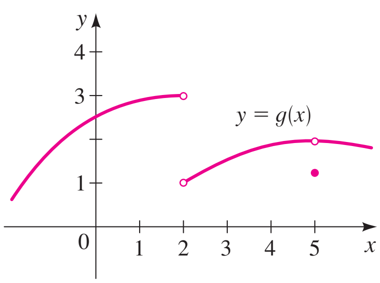


Figure 10

The graph of a function g is shown in Figure 10. Use it to state the values (if they exist) of the following:

(a) $\lim_{x\to 2^-} g(x)$, $\lim_{x\to 2^+} g(x)$, $\lim_{x\to 2} g(x)$

(b) $\lim_{x\to 5^-} g(x)$, $\lim_{x\to 5^+} g(x)$, $\lim_{x\to 5} g(x)$

Solution

(a) From the graph we see that the values of $g(x)$ approach 3 as x approaches 2 from the left, but they approach 1 as x approaches 2 from the right. Therefore

$$\lim_{x\to 2^-} g(x) = 3 \quad \text{and} \quad \lim_{x\to 2^+} g(x) = 1$$

Since the left- and right-hand limits are different, we conclude that $\lim_{x\to 2} g(x)$ does not exist.

(b) The graph also shows that

$$\lim_{x\to 5^-} g(x) = 2 \quad \text{and} \quad \lim_{x\to 5^+} g(x) = 2$$

This time the left- and right-hand limits are the same, and so we have

$$\lim_{x\to 5} g(x) = 2$$

Despite this fact, notice that $g(5) \neq 2$. ■

Example 7 A Piecewise-Defined Function

Let f be the function defined by

$$f(x) = \begin{cases} 2x^2 & \text{if } x < 1 \\ 4 - x & \text{if } x \geq 1 \end{cases}$$

Graph f, and use the graph to find the following:

(a) $\lim_{x\to 1^-} f(x)$ **(b)** $\lim_{x\to 1^+} f(x)$ **(c)** $\lim_{x\to 1} f(x)$

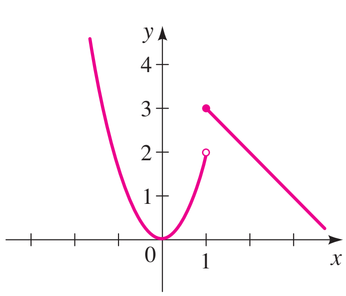


Figure 11

Solution

The graph of f is shown in Figure 11. From the graph we see that the values of $f(x)$ approach 2 as x approaches 1 from the left, but they approach 3 as x approaches 1 from the right. Thus, the left- and right-hand limits are not equal. So we have

(a) $\lim_{x\to 1^-} f(x) = 2$ **(b)** $\lim_{x\to 1^+} f(x) = 3$ **(c)** $\lim_{x\to 1} f(x)$ does not exist. ■

12.1 Exercises

1–6 ■ Complete the table of values (to five decimal places) and use the table to estimate the value of the limit.

1. $\lim\limits_{x\to 4} \dfrac{\sqrt{x} - 2}{x - 4}$

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$						

2. $\lim\limits_{x\to 2} \dfrac{x - 2}{x^2 + x - 6}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

3. $\lim\limits_{x\to 1} \dfrac{x - 1}{x^3 - 1}$

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$						

4. $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

5. $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$

x	± 1	± 0.5	± 0.1	± 0.05	± 0.01
$f(x)$					

6. $\lim\limits_{x\to 0^+} x \ln x$

x	0.1	0.01	0.001	0.0001	0.00001
$f(x)$					

7–12 ■ Use a table of values to estimate the value of the limit. Then use a graphing device to confirm your result graphically.

7. $\lim\limits_{x\to -4} \dfrac{x + 4}{x^2 + 7x + 12}$ **8.** $\lim\limits_{x\to 1} \dfrac{x^3 - 1}{x^2 - 1}$

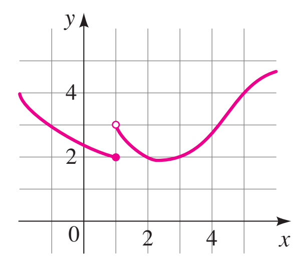
9. $\lim\limits_{x\to 0} \dfrac{5^x - 3^x}{x}$ **10.** $\lim\limits_{x\to 0} \dfrac{\sqrt{x + 9} - 3}{x}$

11. $\lim\limits_{x\to 1} \left(\dfrac{1}{\ln x} - \dfrac{1}{x - 1} \right)$ **12.** $\lim\limits_{x\to 0} \dfrac{\tan 2x}{\tan 3x}$

13. For the function f whose graph is given, state the value of the given quantity, if it exists. If it does not exist, explain why.

(a) $\lim\limits_{x\to 1^-} f(x)$ **(b)** $\lim\limits_{x\to 1^+} f(x)$ **(c)** $\lim\limits_{x\to 1} f(x)$

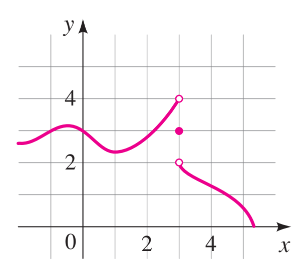
(d) $\lim\limits_{x\to 5} f(x)$ **(e)** $f(5)$



14. For the function f whose graph is given, state the value of the given quantity, if it exists. If it does not exist, explain why.

(a) $\lim\limits_{x\to 0} f(x)$ **(b)** $\lim\limits_{x\to 3^-} f(x)$ **(c)** $\lim\limits_{x\to 3^+} f(x)$

(d) $\lim\limits_{x\to 3} f(x)$ **(e)** $f(3)$

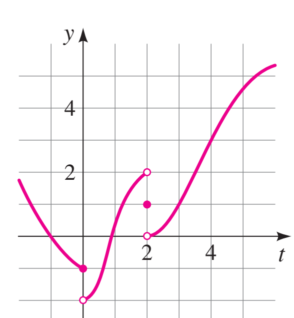


15. For the function g whose graph is given, state the value of the given quantity, if it exists. If it does not exist, explain why.

(a) $\lim\limits_{t\to 0^-} g(t)$ **(b)** $\lim\limits_{t\to 0^+} g(t)$ **(c)** $\lim\limits_{t\to 0} g(t)$

(d) $\lim\limits_{t\to 2^-} g(t)$ **(e)** $\lim\limits_{t\to 2^+} g(t)$ **(f)** $\lim\limits_{t\to 2} g(t)$

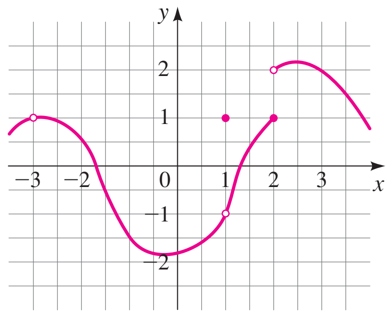
(g) $g(2)$ **(h)** $\lim\limits_{t\to 4} g(t)$



16. State the value of the limit, if it exists, from the given graph of f. If it does not exist, explain why.

(a) $\lim_{x\to 3} f(x)$ **(b)** $\lim_{x\to 1} f(x)$ **(c)** $\lim_{x\to -3} f(x)$

(d) $\lim_{x\to 2^-} f(x)$ **(e)** $\lim_{x\to 2^+} f(x)$ **(f)** $\lim_{x\to 2} f(x)$



17–22 ■ Use a graphing device to determine whether the limit exists. If the limit exists, estimate its value to two decimal places.

17. $\lim_{x\to 1} \dfrac{x^3 + x^2 + 3x - 5}{2x^2 - 5x + 3}$ **18.** $\lim_{x\to 2} \dfrac{x^3 + 6x^2 - 5x + 1}{x^3 - x^2 - 8x + 12}$

19. $\lim_{x\to 0} \ln(\sin^2 x)$ **20.** $\lim_{x\to 0} \dfrac{x^2}{\cos 5x - \cos 4x}$

21. $\lim_{x\to 0} \cos \dfrac{1}{x}$ **22.** $\lim_{x\to 0} \dfrac{1}{1 + e^{1/x}}$

23–26 ■ Graph the piecewise-defined function and use your graph to find the values of the limits, if they exist.

23. $f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ 6 - x & \text{if } x > 2 \end{cases}$

(a) $\lim_{x\to 2^-} f(x)$ **(b)** $\lim_{x\to 2^+} f(x)$ **(c)** $\lim_{x\to 2} f(x)$

24. $f(x) = \begin{cases} 2 & \text{if } x < 0 \\ x + 1 & \text{if } x \ge 0 \end{cases}$

(a) $\lim_{x\to 0^-} f(x)$ **(b)** $\lim_{x\to 0^+} f(x)$ **(c)** $\lim_{x\to 0} f(x)$

25. $f(x) = \begin{cases} -x + 3 & \text{if } x < -1 \\ 3 & \text{if } x \ge -1 \end{cases}$

(a) $\lim_{x\to -1^-} f(x)$ **(b)** $\lim_{x\to -1^+} f(x)$ **(c)** $\lim_{x\to -1} f(x)$

26. $f(x) = \begin{cases} 2x + 10 & \text{if } x \le -2 \\ -x + 4 & \text{if } x > -2 \end{cases}$

(a) $\lim_{x\to -2^-} f(x)$ **(b)** $\lim_{x\to -2^+} f(x)$ **(c)** $\lim_{x\to -2} f(x)$

Discovery • Discussion

27. A Function with Specified Limits Sketch the graph of an example of a function f that satisfies all of the following conditions.

$$\lim_{x\to 0^-} f(x) = 2 \qquad \lim_{x\to 0^+} f(x) = 0$$

$$\lim_{x\to 2} f(x) = 1 \qquad f(0) = 2 \qquad f(2) = 3$$

How many such functions are there?

28. Graphing Calculator Pitfalls

(a) Evaluate $h(x) = (\tan x - x)/x^3$ for $x = 1, 0.5, 0.1, 0.05, 0.01$, and 0.005.

(b) Guess the value of $\lim_{x\to 0} \dfrac{\tan x - x}{x^3}$.

(c) Evaluate $h(x)$ for successively smaller values of x until you finally reach 0 values for $h(x)$. Are you still confident that your guess in part (b) is correct? Explain why you eventually obtained 0 values.

(d) Graph the function h in the viewing rectangle $[-1, 1]$ by $[0, 1]$. Then zoom in toward the point where the graph crosses the y-axis to estimate the limit of $h(x)$ as x approaches 0. Continue to zoom in until you observe distortions in the graph of h. Compare with your results in part (c).

12.2 Finding Limits Algebraically

In Section 12.1 we used calculators and graphs to guess the values of limits, but we saw that such methods don't always lead to the correct answer. In this section, we use algebraic methods to find limits exactly.

Limit Laws

We use the following properties of limits, called the *Limit Laws*, to calculate limits.

Limit Laws

Suppose that c is a constant and that the following limits exist:

$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$

Then

1. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ Limit of a Sum

2. $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$ Limit of a Difference

3. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$ Limit of a Constant Multiple

4. $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$ Limit of a Product

5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$ Limit of a Quotient

These five laws can be stated verbally as follows:

Limit of a Sum **1.** The limit of a sum is the sum of the limits.

Limit of a Difference **2.** The limit of a difference is the difference of the limits.

Limit of a Constant Multiple **3.** The limit of a constant times a function is the constant times the limit of the function.

Limit of a Product **4.** The limit of a product is the product of the limits.

Limit of a Quotient **5.** The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

It's easy to believe that these properties are true. For instance, if $f(x)$ is close to L and $g(x)$ is close to M, it is reasonable to conclude that $f(x) + g(x)$ is close to $L + M$. This gives us an intuitive basis for believing that Law 1 is true.

If we use Law 4 (Limit of a Product) repeatedly with $g(x) = f(x)$, we obtain the following Law 6 for the limit of a power. A similar law holds for roots.

Limit Laws

6. $\lim_{x\to a} [f(x)]^n = [\lim_{x\to a} f(x)]^n$ where n is a positive integer Limit of a Power

7. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer Limit of a Root

[If n is even, we assume that $\lim_{x\to a} f(x) > 0$.]

In words, these laws say:

Limit of a Power **6.** The limit of a power is the power of the limit.

Limit of a Root **7.** The limit of a root is the root of the limit.

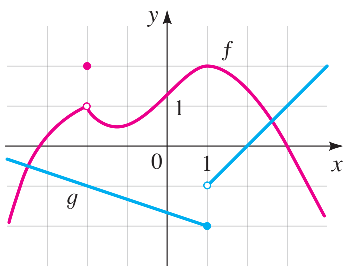


Figure 1

Example 1 Using the Limit Laws

Use the Limit Laws and the graphs of f and g in Figure 1 to evaluate the following limits, if they exist.

(a) $\lim_{x \to -2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 1} [f(x)g(x)]$

(c) $\lim_{x \to 2} \dfrac{f(x)}{g(x)}$ (d) $\lim_{x \to 1} [f(x)]^3$

Solution

(a) From the graphs of f and g we see that

$$\lim_{x \to -2} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to -2} g(x) = -1$$

Therefore, we have

$$\lim_{x \to -2} [f(x) + 5g(x)] = \lim_{x \to -2} f(x) + \lim_{x \to -2} [5g(x)] \qquad \text{Limit of a Sum}$$

$$= \lim_{x \to -2} f(x) + 5 \lim_{x \to -2} g(x) \qquad \text{Limit of a Constant Multiple}$$

$$= 1 + 5(-1) = -4$$

(b) We see that $\lim_{x \to 1} f(x) = 2$. But $\lim_{x \to 1} g(x)$ does not exist because the left- and right-hand limits are different:

$$\lim_{x \to 1^-} g(x) = -2 \qquad \lim_{x \to 1^+} g(x) = -1$$

So we can't use Law 4 (Limit of a Product). The given limit does not exist, since the left-hand limit is not equal to the right-hand limit.

(c) The graphs show that

$$\lim_{x \to 2} f(x) \approx 1.4 \qquad \text{and} \qquad \lim_{x \to 2} g(x) = 0$$

Because the limit of the denominator is 0, we can't use Law 5 (Limit of a Quotient). The given limit does not exist because the denominator approaches 0 while the numerator approaches a nonzero number.

(d) Since $\lim_{x \to 1} f(x) = 2$, we use Law 6 to get

$$\lim_{x \to 1} [f(x)]^3 = [\lim_{x \to 1} f(x)]^3 \qquad \text{Limit of a Power}$$

$$= 2^3 = 8$$

■

Applying the Limit Laws

In applying the Limit Laws, we need to use four special limits.

Some Special Units

1. $\lim_{x \to a} c = c$
2. $\lim_{x \to a} x = a$
3. $\lim_{x \to a} x^n = a^n$ where n is a positive integer
4. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer and $a > 0$

Special Limits 1 and 2 are intuitively obvious—looking at the graphs of $y = c$ and $y = x$ will convince you of their validity. Limits 3 and 4 are special cases of Limit Laws 6 and 7 (Limits of a Power and of a Root).

Example 2 Using the Limit Laws

Evaluate the following limits and justify each step.

(a) $\lim_{x\to 5}(2x^2 - 3x + 4)$ (b) $\lim_{x\to -2}\dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

Solution

(a)

$$\lim_{x\to 5}(2x^2 - 3x + 4) = \lim_{x\to 5}(2x^2) - \lim_{x\to 5}(3x) + \lim_{x\to 5}4$$ Limits of a Difference and Sum

$$= 2\lim_{x\to 5}x^2 - 3\lim_{x\to 5}x + \lim_{x\to 5}4$$ Limit of a Constant Multiple

$$= 2(5^2) - 3(5) + 4$$ Special Limits 3, 2, and 1

$$= 39$$

(b) We start by using Law 5, but its use is fully justified only at the final stage when we see that the limits of the numerator and denominator exist and the limit of the denominator is not 0.

$$\lim_{x\to -2}\frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{\lim_{x\to -2}(x^3 + 2x^2 - 1)}{\lim_{x\to -2}(5 - 3x)}$$ Limit of a Quotient

$$= \frac{\lim_{x\to -2}x^3 + 2\lim_{x\to -2}x^2 - \lim_{x\to -2}1}{\lim_{x\to -2}5 - 3\lim_{x\to -2}x}$$ Limits of Sums, Differences, and Constant Multiples

$$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)}$$ Special Limits 3, 2, and 1

$$= -\frac{1}{11}$$ ■

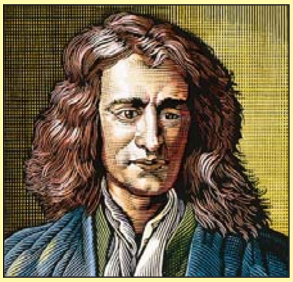
If we let $f(x) = 2x^2 - 3x + 4$, then $f(5) = 39$. In Example 2(a), we found that $\lim_{x\to 5} f(x) = 39$. In other words, we would have gotten the correct answer by substituting 5 for x. Similarly, direct substitution provides the correct answer in part (b). The functions in Example 2 are a polynomial and a rational function, respectively, and similar use of the Limit Laws proves that direct substitution always works for such functions. We state this fact as follows.

Limits by Direct Substitution

If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)$$

Functions with this direct substitution property are called **continuous at** a. You will learn more about continuous functions when you study calculus.



Bill Sanederson/SPL/Photo Researchers, Inc.

Sir Isaac Newton (1642–1727) is universally regarded as one of the giants of physics and mathematics. He is well known for discovering the laws of motion and gravity and for inventing the calculus, but he also proved the Binomial Theorem and the laws of optics, and developed methods for solving polynomial equations to any desired accuracy. He was born on Christmas Day, a few months after the death of his father. After an unhappy childhood, he entered Cambridge University, where he learned mathematics by studying the writings of Euclid and Descartes.

During the plague years of 1665 and 1666, when the university was closed, Newton thought and wrote about ideas that, once published, instantly revolutionized the sciences. Imbued with a pathological fear of criticism, he published these writings only after many years of encouragement from Edmund Halley (who discovered the now-famous comet) and other colleagues.

Newton's works brought him enormous fame and prestige. Even poets were moved to praise; Alexander Pope wrote:

> Nature and Nature's Laws
> lay hid in Night.
> God said, "Let Newton be"
> and all was Light.

(continued)

Example 3 Finding Limits by Direct Substitution

Evaluate the following limits.

(a) $\lim_{x \to 3} (2x^3 - 10x - 8)$ (b) $\lim_{x \to -1} \dfrac{x^2 + 5x}{x^4 + 2}$

Solution

(a) The function $f(x) = 2x^3 - 10x - 12$ is a polynomial, so we can find the limit by direct substitution:

$$\lim_{x \to 3} (2x^3 - 10x - 12) = 2(3)^3 - 10(3) - 8 = 16$$

(b) The function $f(x) = (x^2 + 5x)/(x^4 + 2)$ is a rational function, and $x = -1$ is in its domain (because the denominator is not zero for $x = -1$). Thus, we can find the limit by direct substitution:

$$\lim_{x \to -1} \frac{x^2 + 5x}{x^4 + 2} = \frac{(-1)^2 + 5(-1)}{(-1)^4 + 2} = -\frac{4}{3}$$

■

Finding Limits Using Algebra and the Limit Laws

As we saw in Example 3, evaluating limits by direct substitution is easy. But not all limits can be evaluated this way. In fact, most of the situations in which limits are useful requires us to work harder to evaluate the limit. The next three examples illustrate how we can use algebra to find limits.

Example 4 Finding a Limit by Canceling a Common Factor

Find $\lim_{x \to 1} \dfrac{x - 1}{x^2 - 1}$.

Solution Let $f(x) = (x - 1)/(x^2 - 1)$. We can't find the limit by substituting $x = 1$ because $f(1)$ isn't defined. Nor can we apply Law 5 (Limit of a Quotient) because the limit of the denominator is 0. Instead, we need to do some preliminary algebra. We factor the denominator as a difference of squares:

$$\frac{x - 1}{x^2 - 1} = \frac{x - 1}{(x - 1)(x + 1)}$$

The numerator and denominator have a common factor of $x - 1$. When we take the limit as x approaches 1, we have $x \neq 1$ and so $x - 1 \neq 0$. Therefore, we can cancel the common factor and compute the limit as follows:

$$\lim_{x \to 1} \frac{x - 1}{x^2 - 1} = \lim_{x \to 1} \frac{x - 1}{(x - 1)(x + 1)} \qquad \text{Factor}$$

$$= \lim_{x \to 1} \frac{1}{x + 1} \qquad \text{Cancel}$$

$$= \frac{1}{1 + 1} = \frac{1}{2} \qquad \text{Let } x \to 1$$

This calculation confirms algebraically the answer we got numerically and graphically in Example 1 in Section 12.1. ■

Newton was far more modest about his accomplishments. He said, "I seem to have been only like a boy playing on the seashore . . . while the great ocean of truth lay all undiscovered before me." Newton was knighted by Queen Anne in 1705 and was buried with great honor in Westminster Abbey.

Example 5 Finding a Limit by Simplifying

Evaluate $\lim_{h\to 0} \dfrac{(3+h)^2 - 9}{h}$.

Solution We can't use direct substitution to evaluate this limit, because the limit of the denominator is 0. So we first simplify the limit algebraically.

$$\begin{aligned}
\lim_{h\to 0} \frac{(3+h)^2 - 9}{h} &= \lim_{h\to 0} \frac{(9 + 6h + h^2) - 9}{h} && \text{Expand} \\
&= \lim_{h\to 0} \frac{6h + h^2}{h} && \text{Simplify} \\
&= \lim_{h\to 0} (6 + h) && \text{Cancel } h \\
&= 6 && \text{Let } h \to 0
\end{aligned}$$

■

Example 6 Finding a Limit by Rationalizing

Find $\lim_{t\to 0} \dfrac{\sqrt{t^2+9} - 3}{t^2}$.

Solution We can't apply Law 5 (Limit of a Quotient) immediately, since the limit of the denominator is 0. Here the preliminary algebra consists of rationalizing the numerator:

$$\begin{aligned}
\lim_{t\to 0} \frac{\sqrt{t^2+9} - 3}{t^2} &= \lim_{t\to 0} \frac{\sqrt{t^2+9} - 3}{t^2} \cdot \frac{\sqrt{t^2+9} + 3}{\sqrt{t^2+9} + 3} && \text{Rationalize numerator} \\
&= \lim_{t\to 0} \frac{(t^2+9) - 9}{t^2\left(\sqrt{t^2+9} + 3\right)} = \lim_{t\to 0} \frac{t^2}{t^2\left(\sqrt{t^2+9} + 3\right)} \\
&= \lim_{t\to 0} \frac{1}{\sqrt{t^2+9} + 3} = \frac{1}{\sqrt{\lim_{t\to 0}(t^2+9)} + 3} = \frac{1}{3+3} = \frac{1}{6}
\end{aligned}$$

This calculation confirms the guess that we made in Example 2 in Section 12.1. ■

Using Left- and Right-Hand Limits

Some limits are best calculated by first finding the left- and right-hand limits. The following theorem is a reminder of what we discovered in Section 12.1. It says that *a two-sided limit exists if and only if both of the one-sided limits exist and are equal*.

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

When computing one-sided limits, we use the fact that the Limit Laws also hold for one-sided limits.

Example 7 Comparing Right and Left Limits

Show that $\lim_{x\to 0} |x| = 0$.

Solution Recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Since $|x| = x$ for $x > 0$, we have

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$$

For $x < 0$, we have $|x| = -x$ and so

$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0$$

Therefore

$$\lim_{x\to 0} |x| = 0$$

■

The result of Example 7 looks plausible from Figure 2.

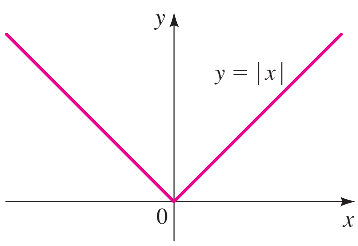


Figure 2

Example 8 Comparing Right and Left Limits

Prove that $\lim_{x\to 0} \dfrac{|x|}{x}$ does not exist.

Solution Since $|x| = x$ for $x > 0$ and $|x| = -x$ for $x < 0$, we have

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$$

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} (-1) = -1$$

Since the right-hand and left-hand limits exist and are different, it follows that $\lim_{x\to 0} |x|/x$ does not exist. The graph of the function $f(x) = |x|/x$ is shown in Figure 3 and supports the limits that we found. ■

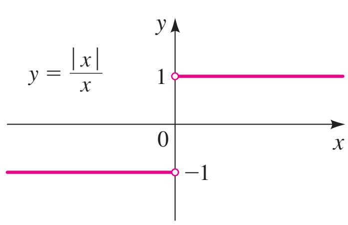


Figure 3

Example 9 The Limit of a Piecewise-Defined Function

Let

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8 - 2x & \text{if } x < 4 \end{cases}$$

Determine whether $\lim_{x\to 4} f(x)$ exists.

Solution Since $f(x) = \sqrt{x-4}$ for $x > 4$, we have

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \sqrt{x-4} = \sqrt{4-4} = 0$$

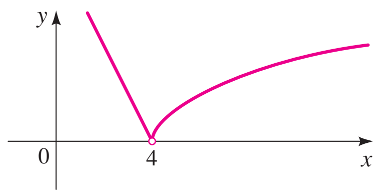


Figure 4

Since $f(x) = 8 - 2x$ for $x < 4$, we have

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-}(8 - 2x) = 8 - 2\cdot 4 = 0$$

The right- and left-hand limits are equal. Thus, the limit exists and

$$\lim_{x\to 4} f(x) = 0$$

The graph of f is shown in Figure 4. ■

12.2 Exercises

1. Suppose that

$$\lim_{x\to a} f(x) = -3 \qquad \lim_{x\to a} g(x) = 0 \qquad \lim_{x\to a} h(x) = 8$$

Find the value of the given limit. If the limit does not exist, explain why.

(a) $\lim_{x\to a}[f(x) + h(x)]$ **(b)** $\lim_{x\to a}[f(x)]^2$

(c) $\lim_{x\to a}\sqrt[3]{h(x)}$ **(d)** $\lim_{x\to a}\dfrac{1}{f(x)}$

(e) $\lim_{x\to a}\dfrac{f(x)}{h(x)}$ **(f)** $\lim_{x\to a}\dfrac{g(x)}{f(x)}$

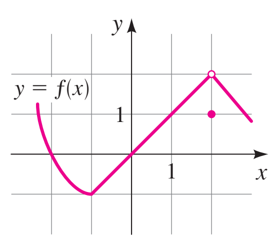
(g) $\lim_{x\to a}\dfrac{f(x)}{g(x)}$ **(h)** $\lim_{x\to a}\dfrac{2f(x)}{h(x) - f(x)}$

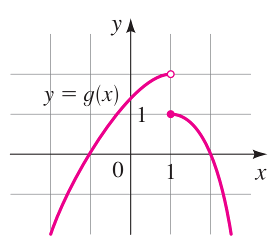
2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.

(a) $\lim_{x\to 2}[f(x) + g(x)]$ **(b)** $\lim_{x\to 1}[f(x) + g(x)]$

(c) $\lim_{x\to 0}[f(x)g(x)]$ **(d)** $\lim_{x\to -1}\dfrac{f(x)}{g(x)}$

(e) $\lim_{x\to 2} x^3 f(x)$ **(f)** $\lim_{x\to 1}\sqrt{3 + f(x)}$





3–8 ■ Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

3. $\lim_{x\to 4}(5x^2 - 2x + 3)$

4. $\lim_{x\to 3}(x^3 + 2)(x^2 - 5x)$

5. $\lim_{x\to -1}\dfrac{x - 2}{x^2 + 4x - 3}$

6. $\lim_{x\to 1}\left(\dfrac{x^4 + x^2 - 6}{x^4 + 2x + 3}\right)^2$

7. $\lim_{t\to -2}(t + 1)^9(t^2 - 1)$

8. $\lim_{u\to -2}\sqrt{u^4 + 3u + 6}$

9–20 ■ Evaluate the limit, if it exists.

9. $\lim_{x\to 2}\dfrac{x^2 + x - 6}{x - 2}$

10. $\lim_{x\to -4}\dfrac{x^2 + 5x + 4}{x^2 + 3x - 4}$

11. $\lim_{x\to 2}\dfrac{x^2 - x + 6}{x + 2}$

12. $\lim_{x\to 1}\dfrac{x^3 - 1}{x^2 - 1}$

13. $\lim_{t\to -3}\dfrac{t^2 - 9}{2t^2 + 7t + 3}$

14. $\lim_{h\to 0}\dfrac{\sqrt{1 + h} - 1}{h}$

15. $\lim_{h\to 0}\dfrac{(2 + h)^3 - 8}{h}$

16. $\lim_{x\to 2}\dfrac{x^4 - 16}{x - 2}$

17. $\lim_{x\to 7}\dfrac{\sqrt{x + 2} - 3}{x - 7}$

18. $\lim_{h\to 0}\dfrac{(3 + h)^{-1} - 3^{-1}}{h}$

19. $\lim_{x\to -4}\dfrac{\frac{1}{4} + \frac{1}{x}}{4 + x}$

20. $\lim_{t\to 0}\left(\dfrac{1}{t} - \dfrac{1}{t^2 + t}\right)$

21–24 ■ Find the limit and use a graphing device to confirm your result graphically.

21. $\lim_{x\to 1}\dfrac{x^2 - 1}{\sqrt{x} - 1}$

22. $\lim_{x\to 0}\dfrac{(4 + x)^3 - 64}{x}$

23. $\lim_{x\to -1}\dfrac{x^2 - x - 2}{x^3 - x}$

24. $\lim_{x\to 1}\dfrac{x^8 - 1}{x^5 - x}$

25. **(a)** Estimate the value of

$$\lim_{x\to 0}\frac{x}{\sqrt{1 + 3x} - 1}$$

by graphing the function $f(x) = x/\left(\sqrt{1 + 3x} - 1\right)$.

(b) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.

(c) Use the Limit Laws to prove that your guess is correct.

26. **(a)** Use a graph of

$$f(x) = \frac{\sqrt{3 + x} - \sqrt{3}}{x}$$

to estimate the value of $\lim_{x\to 0} f(x)$ to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Use the Limit Laws to find the exact value of the limit.

27–32 ■ Find the limit, if it exists. If the limit does not exist, explain why.

27. $\displaystyle\lim_{x\to -4} |x + 4|$

28. $\displaystyle\lim_{x\to -4^-} \frac{|x + 4|}{x + 4}$

29. $\displaystyle\lim_{x\to 2} \frac{|x - 2|}{x - 2}$

30. $\displaystyle\lim_{x\to 1.5} \frac{2x^2 - 3x}{|2x - 3|}$

31. $\displaystyle\lim_{x\to 0^-} \left(\frac{1}{x} - \frac{1}{|x|}\right)$

32. $\displaystyle\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{|x|}\right)$

33. Let

$$f(x) = \begin{cases} x - 1 & \text{if } x < 2 \\ x^2 - 4x + 6 & \text{if } x \geq 2 \end{cases}$$

(a) Find $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$.

(b) Does $\lim_{x\to 2} f(x)$ exist?

(c) Sketch the graph of f.

34. Let

$$h(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } 0 < x \leq 2 \\ 8 - x & \text{if } x > 2 \end{cases}$$

(a) Evaluate each limit, if it exists.

(i) $\displaystyle\lim_{x\to 0^+} h(x)$ (iv) $\displaystyle\lim_{x\to 2^-} h(x)$

(ii) $\displaystyle\lim_{x\to 0} h(x)$ (v) $\displaystyle\lim_{x\to 2^+} h(x)$

(iii) $\displaystyle\lim_{x\to 1} h(x)$ (vi) $\displaystyle\lim_{x\to 2} h(x)$

(b) Sketch the graph of h.

Discovery • Discussion

35. Cancellation and Limits

(a) What is wrong with the following equation?

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

(b) In view of part (a), explain why the equation

$$\lim_{x\to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x\to 2} (x + 3)$$

is correct.

36. The Lorentz Contraction In the theory of relativity, the Lorentz contraction formula

$$L = L_0\sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v\to c^-} L$ and interpret the result. Why is a left-hand limit necessary?

37. Limits of Sums and Products

(a) Show by means of an example that $\lim_{x\to a} [f(x) + g(x)]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

(b) Show by means of an example that $\lim_{x\to a} [f(x)g(x)]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

12.3 Tangent Lines and Derivatives

In this section we see how limits arise when we attempt to find the tangent line to a curve or the instantaneous rate of change of a function.

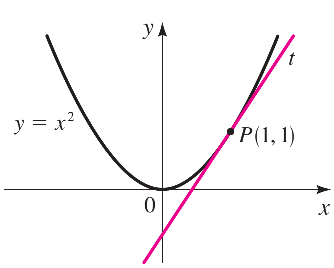


Figure 1

The Tangent Problem

A *tangent line* is a line that *just* touches a curve. For instance, Figure 1 shows the parabola $y = x^2$ and the tangent line t that touches the parabola at the point $P(1, 1)$. We will be able to find an equation of the tangent line t as soon as we know its slope m. The difficulty is that we know only one point, P, on t, whereas we need two points to compute the slope. But observe that we can compute an approximation to m by

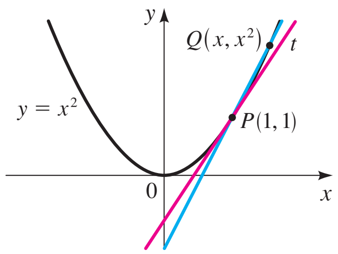


Figure 2

choosing a nearby point $Q(x, x^2)$ on the parabola (as in Figure 2) and computing the slope m_{PQ} of the secant line PQ.

We choose $x \neq 1$ so that $Q \neq P$. Then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

Now we let x approach 1, so Q approaches P along the parabola. Figure 3 shows how the corresponding secant lines rotate about P and approach the tangent line t.

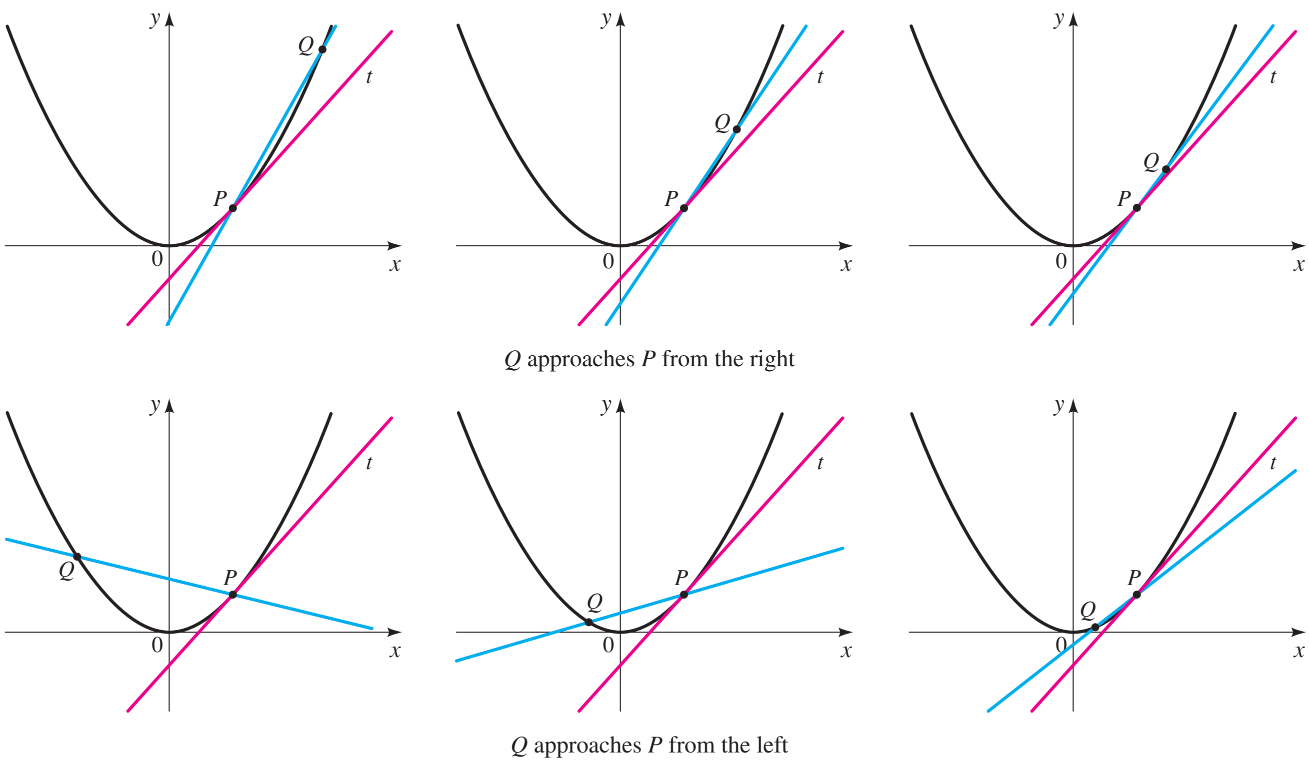


Figure 3

The slope of the tangent line is the limit of the slopes of the secant lines:

$$m = \lim_{Q \to P} m_{PQ}$$

So, using the method of Section 12.2, we have

$$m = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1}$$

$$= \lim_{x \to 1} (x + 1) = 1 + 1 = 2$$

The point-slope form for the equation of a line through the point (x_1, y_1) with slope m is

$$y - y_1 = m(x - x_1)$$

(See Section 1.10.)

Now that we know the slope of the tangent line is $m = 2$, we can use the point-slope form of the equation of a line to find its equation:

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$

We sometimes refer to the slope of the tangent line to a curve at a point as the **slope of the curve** at the point. The idea is that if we zoom in far enough toward the point, the curve looks almost like a straight line. Figure 4 illustrates this procedure for the curve $y = x^2$. The more we zoom in, the more the parabola looks like a line. In other words, the curve becomes almost indistinguishable from its tangent line.

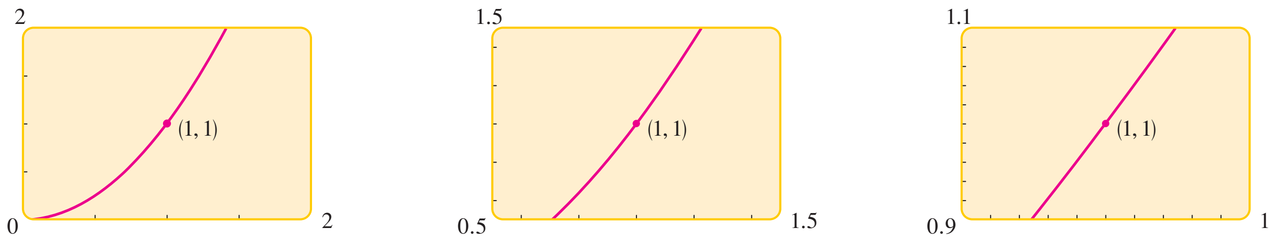


Figure 4
Zooming in toward the point (1, 1) on the parabola $y = x^2$

If we have a general curve C with equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Then we let Q approach P along the curve C by letting x approach a. If m_{PQ} approaches a number m, then we define the *tangent* t to be the line through P with slope m. (This amounts to saying that the tangent line is the limiting position of the secant line PQ as Q approaches P. See Figure 5.)

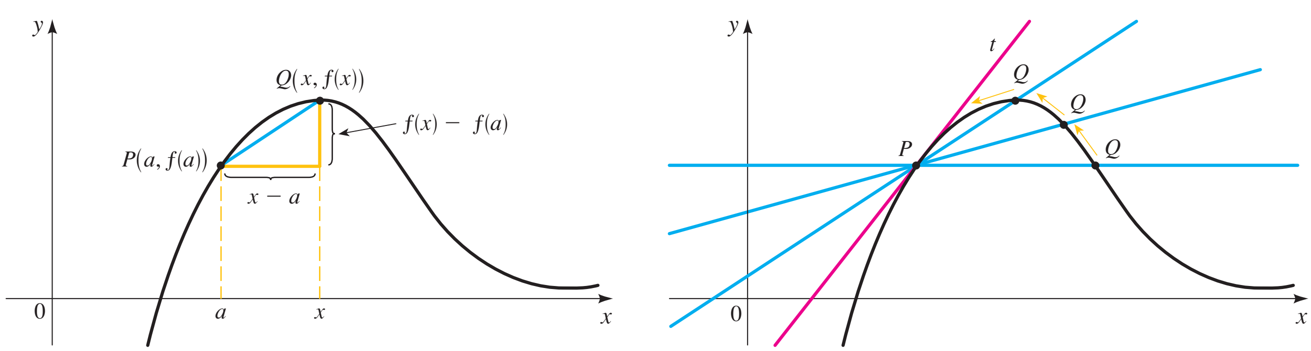


Figure 5

Definition of a Tangent Line

The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Example 1 Finding a Tangent Line to a Hyperbola

Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point $(3, 1)$.

Solution Let $f(x) = 3/x$. Then the slope of the tangent line at $(3, 1)$ is

$$m = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} \qquad \text{Definition of } m$$

$$= \lim_{x \to 3} \frac{\dfrac{3}{x} - 1}{x - 3} \qquad f(x) = \frac{3}{x}$$

$$= \lim_{x \to 3} \frac{3 - x}{x(x - 3)} \qquad \text{Multiply numerator and denominator by } x$$

$$= \lim_{x \to 3} \left(-\frac{1}{x} \right) \qquad \text{Cancel } x - 3$$

$$= -\frac{1}{3} \qquad \text{Let } x \to 3$$

Therefore, an equation of the tangent at the point $(3, 1)$ is

$$y - 1 = -\tfrac{1}{3}(x - 3)$$

which simplifies to

$$x + 3y - 6 = 0$$

The hyperbola and its tangent are shown in Figure 6. ■

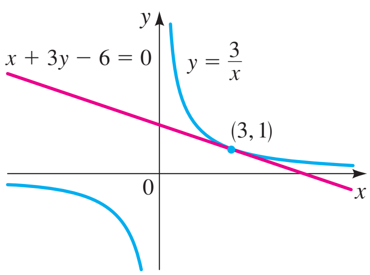


Figure 6

There is another expression for the slope of a tangent line that is sometimes easier to use. Let $h = x - a$. Then $x = a + h$, so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a + h) - f(a)}{h}$$

See Figure 7 where the case $h > 0$ is illustrated and Q is to the right of P. If it happened that $h < 0$, however, Q would be to the left of P.

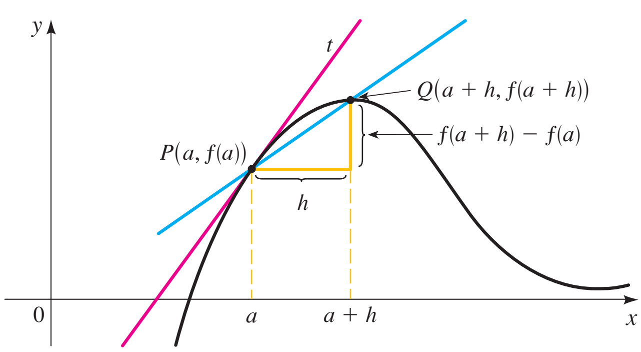


Figure 7

Notice that as x approaches a, h approaches 0 (because $h = x - a$), and so the expression for the slope of the tangent line becomes

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Newton and Limits

In 1687 Isaac Newton (see page 894) published his masterpiece *Principia Mathematica*. In this work, the greatest scientific treatise ever written, Newton set forth his version of calculus and used it to investigate mechanics, fluid dynamics, and wave motion, and to explain the motion of planets and comets.

The beginnings of calculus are found in the calculations of areas and volumes by ancient Greek scholars such as Eudoxus and Archimedes. Although aspects of the idea of a limit are implicit in their "method of exhaustion," Eudoxus and Archimedes never explicitly formulated the concept of a limit. Likewise, mathematicians such as Cavalieri, Ferinat, and Barrow, the immediate precursors of Newton in the development of calculus, did not actually use limits. It was Isaac Newton who first talked explicitly about limits. He explained that the main idea behind limits is that quantities "approach nearer than by any given difference." Newton stated that the limit was the basic concept in calculus but it was left to later mathematicians like Cauchy to clarify these ideas.

Example 2 Finding a Tangent Line

Find an equation of the tangent line to the curve $y = x^3 - 2x + 3$ at the point $(1, 2)$.

Solution If $f(x) = x^3 - 2x + 3$, then the slope of the tangent line where $a = 1$ is

$$
\begin{aligned}
m &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} && \text{Definition of } m \\
&= \lim_{h\to 0} \frac{[(1+h)^3 - 2(1+h) + 3] - [1^3 - 2(1) + 3]}{h} && f(x) = x^3 - 2x + 3 \\
&= \lim_{h\to 0} \frac{1 + 3h + 3h^2 + h^3 - 2 - 2h + 3 - 2}{h} && \text{Expand numerator} \\
&= \lim_{h\to 0} \frac{h + 3h^2 + h^3}{h} && \text{Simplify} \\
&= \lim_{h\to 0} (1 + 3h + h^2) && \text{Cancel } h \\
&= 1 && \text{Let } h \to 0
\end{aligned}
$$

So an equation of the tangent line at $(1, 2)$ is

$$y - 2 = 1(x - 1) \quad \text{or} \quad y = x + 1$$

■

Derivatives

We have seen that the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ can be written as

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

It turns out that this expression arises in many other contexts as well, such as finding velocities and other rates of change. Because this type of limit occurs so widely, it is given a special name and notation.

Definition of a Derivative

The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

Example 3 Finding a Derivative at a Point

Find the derivative of the function $f(x) = 5x^2 + 3x - 1$ at the number 2.

Solution According to the definition of a derivative, with $a = 2$, we have

$$\begin{aligned}
f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} && \text{Definition of } f'(2) \\
&= \lim_{h \to 0} \frac{[5(2+h)^2 + 3(2+h) - 1] - [5(2)^2 + 3(2) - 1]}{h} && f(x) = 5x^2 + 3x - 1 \\
&= \lim_{h \to 0} \frac{20 + 20h + 5h^2 + 6 + 3h - 1 - 25}{h} && \text{Expand} \\
&= \lim_{h \to 0} \frac{23h + 5h^2}{h} && \text{Simplify} \\
&= \lim_{h \to 0} (23 + 5h) && \text{Cancel } h \\
&= 23 && \text{Let } h \to 0
\end{aligned}$$

■

We see from the definition of a derivative that the number $f'(a)$ is the same as the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$. So the result of Example 2 shows that the slope of the tangent line to the parabola $y = 5x^2 + 3x - 1$ at the point $(2, 25)$ is $f'(2) = 23$.

Example 4 Finding a Derivative

Let $f(x) = \sqrt{x}$.

(a) Find $f'(a)$.

(b) Find $f'(1)$, $f'(4)$, and $f'(9)$.

Solution

(a) We use the definition of the derivative at a:

$$\begin{aligned}
f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} && \text{Definition of derivative} \\
&= \lim_{h \to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h} && f(x) = \sqrt{x} \\
&= \lim_{h \to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h} \cdot \frac{\sqrt{a+h} + \sqrt{a}}{\sqrt{a+h} + \sqrt{a}} && \text{Rationalize numerator} \\
&= \lim_{h \to 0} \frac{(a+h) - a}{h\left(\sqrt{a+h} + \sqrt{a}\right)} && \text{Difference of squares} \\
&= \lim_{h \to 0} \frac{h}{h\left(\sqrt{a+h} + \sqrt{a}\right)} && \text{Simplify numerator}
\end{aligned}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{a+h} + \sqrt{a}}$$ Cancel h

$$= \frac{1}{\sqrt{a} + \sqrt{a}} = \frac{1}{2\sqrt{a}}$$ Let $h \to 0$

(b) Substituting $a = 1$, $a = 4$, and $a = 9$ into the result of part (a), we get

$$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2} \qquad f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4} \qquad f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

These values of the derivative are the slopes of the tangent lines shown in Figure 8.

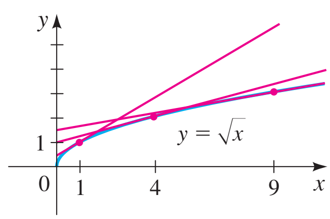


Figure 8

Instantaneous Rates of Change

In Section 2.3 we defined the average rate of change of a function f between the numbers a and x as

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(x) - f(a)}{x - a}$$

Suppose we consider the average rate of change over smaller and smaller intervals by letting x approach a. The limit of these average rates of change is called the instantaneous rate of change.

Instantaneous Rate of Change

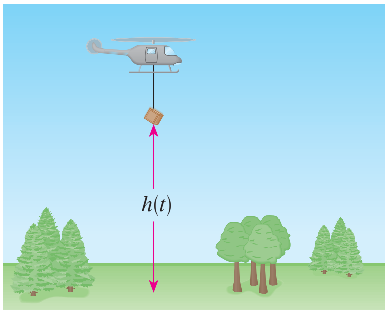
If $y = f(x)$, the **instantaneous rate of change of y with respect to x** at $x = a$ is the limit of the average rates of change as x approaches a:

$$\text{instantaneous rate of change} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

Notice that we now have two ways of interpreting the derivative:

- $f'(a)$ is the slope of the tangent line to $y = f(x)$ at $x = a$
- $f'(a)$ is the instantaneous rate of change of y with respect to x at $x = a$

In the special case where $x = t = \text{time}$ and $s = f(t) = \text{displacement}$ (directed distance) at time t of an object traveling in a straight line, the instantaneous rate of change is called the **instantaneous velocity**.



Example 5 Instantaneous Velocity of a Falling Object

If an object is dropped from a height of 3000 ft, its distance above the ground (in feet) after t seconds is given by $h(t) = 3000 - 16t^2$. Find the object's instantaneous velocity after 4 seconds.

Solution After 4 s have elapsed, the height is $h(4) = 2744$ ft. The instantaneous velocity is

$$\begin{aligned} h'(4) &= \lim_{t\to4} \frac{h(t) - h(4)}{t - 4} && \text{Definition of } h'(4) \\ &= \lim_{t\to4} \frac{3000 - 16t^2 - 2744}{t - 4} && h(t) = 3000 - 16t^2 \\ &= \lim_{t\to4} \frac{256 - 16t^2}{t - 4} && \text{Simplify} \\ &= \lim_{t\to4} \frac{16(4 - t)(4 + t)}{t - 4} && \text{Factor numerator} \\ &= \lim_{t\to4} -16(4 + t) && \text{Cancel } t - 4 \\ &= -16(4 + 4) = -128 \text{ ft/s} && \text{Let } t \to 4 \end{aligned}$$

The negative sign indicates that the height is *decreasing* at a rate of 128 ft/s. ■

t	$P(t)$
1996	269,667,000
1998	276,115,000
2000	282,192,000
2002	287,941,000
2004	293,655,000

t	$\dfrac{P(t) - P(2000)}{t - 2000}$
1996	3,131,250
1998	3,038,500
2002	2,874,500
2004	2,865,750

Here we have estimated the derivative by averaging the slopes of two secant lines. Another method is to plot the population function and estimate the slope of the tangent line when $t = 2000$.

Example 6 Estimating an Instantaneous Rate of Change

Let $P(t)$ be the population of the United States at time t. The table in the margin gives approximate values of this function by providing midyear population estimates from 1996 to 2004. Interpret and estimate the value of $P'(2000)$.

Solution The derivative $P'(2000)$ means the rate of change of P with respect to t when $t = 2000$, that is, the rate of increase of the population in 2000.

According to the definition of a derivative, we have

$$P'(2000) = \lim_{t\to2000} \frac{P(t) - P(2000)}{t - 2000}$$

So we compute and tabulate values of the difference quotient (the average rates of change) as shown in the table in the margin. We see that $P'(2000)$ lies somewhere between 3,038,500 and 2,874,500. (Here we are making the reasonable assumption that the population didn't fluctuate wildly between 1996 and 2004.) We estimate that the rate of increase of the U.S. population in 2000 was the average of these two numbers, namely

$$P'(2000) \approx 2.96 \text{ million people/year}$$ ■

12.3 Exercises

1–6 ■ Find the slope of the tangent line to the graph of f at the given point.

1. $f(x) = 3x + 4$ at $(1, 7)$

2. $f(x) = 5 - 2x$ at $(-3, 11)$

3. $f(x) = 4x^2 - 3x$ at $(-1, 7)$

4. $f(x) = 1 + 2x - 3x^2$ at $(1, 0)$

5. $f(x) = 2x^3$ at $(2, 16)$

6. $f(x) = \dfrac{6}{x + 1}$ at $(2, 2)$

7–12 ■ Find an equation of the tangent line to the curve at the given point. Graph the curve and the tangent line.

7. $y = x + x^2$ at $(-1, 0)$

8. $y = 2x - x^3$ at $(1, 1)$

9. $y = \dfrac{x}{x - 1}$ at $(2, 2)$

10. $y = \dfrac{1}{x^2}$ at $(-1, 1)$

11. $y = \sqrt{x + 3}$ at $(1, 2)$

12. $y = \sqrt{1 + 2x}$ at $(4, 3)$

13–18 ■ Find the derivative of the function at the given number.

13. $f(x) = 1 - 3x^2$ at 2

14. $f(x) = 2 - 3x + x^2$ at -1

15. $g(x) = x^4$ at 1

16. $g(x) = 2x^2 + x^3$ at 1

17. $F(x) = \dfrac{1}{\sqrt{x}}$ at 4

18. $G(x) = 1 + 2\sqrt{x}$ at 4

19–22 ■ Find $f'(a)$, where a is in the domain of f.

19. $f(x) = x^2 + 2x$

20. $f(x) = -\dfrac{1}{x^2}$

21. $f(x) = \dfrac{x}{x + 1}$

22. $f(x) = \sqrt{x - 2}$

23. **(a)** If $f(x) = x^3 - 2x + 4$, find $f'(a)$.

(b) Find equations of the tangent lines to the graph of f at the points whose x-coordinates are 0, 1, and 2.

(c) Graph f and the three tangent lines.

24. **(a)** If $g(x) = 1/(2x - 1)$, find $g'(a)$.

(b) Find equations of the tangent lines to the graph of g at the points whose x-coordinates are -1, 0, and 1.

(c) Graph g and the three tangent lines.

Applications

25. Velocity of a Ball If a ball is thrown into the air with a velocity of 40 ft/s, its height (in feet) after t seconds is given by $y = 40t - 16t^2$. Find the velocity when $t = 2$.

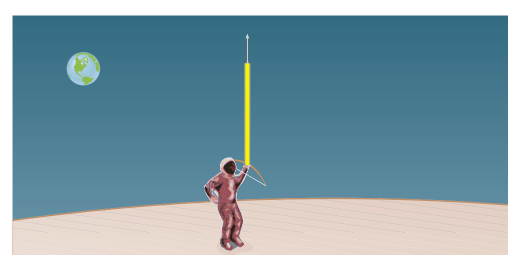
26. Velocity on the Moon If an arrow is shot upward on the moon with a velocity of 58 m/s, its height (in meters) after t seconds is given by $H = 58t - 0.83t^2$.

(a) Find the velocity of the arrow after one second.

(b) Find the velocity of the arrow when $t = a$.

(c) At what time t will the arrow hit the moon?

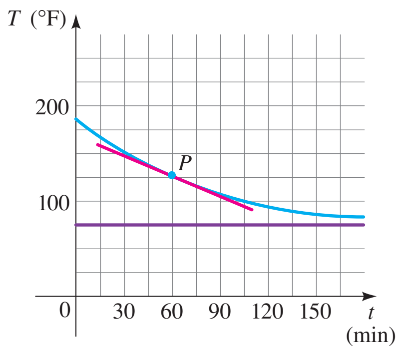
(d) With what velocity will the arrow hit the moon?



27. Velocity of a Particle The displacement s (in meters) of a particle moving in a straight line is given by the equation of motion $s = 4t^3 + 6t + 2$, where t is measured in seconds. Find the velocity of the particle s at times $t = a$, $t = 1$, $t = 2$, $t = 3$.

28. Inflating a Balloon A spherical balloon is being inflated. Find the rate of change of the surface area $(S = 4\pi r^2)$ with respect to the radius r when $r = 2$ ft.

29. Temperature Change A roast turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F. The graph shows how the temperature of the turkey decreases and eventually approaches room temperature. By measuring the slope of the tangent, estimate the rate of change of the temperature after an hour.



30. Heart Rate A cardiac monitor is used to measure the heart rate of a patient after surgery. It compiles the number of heartbeats after t minutes. When the data in the table are graphed, the slope of the tangent line represents the heart rate in beats per minute.

t (min)	36	38	40	42	44
Heartbeats	2530	2661	2806	2948	3080

(a) Find the average heart rates (slopes of the secant lines) over the time intervals $[40, 42]$ and $[42, 44]$.

(b) Estimate the patient's heart rate after 42 minutes by averaging the slopes of these two secant lines.

31. Water Flow A tank holds 1000 gallons of water, which drains from the bottom of the tank in half an hour. The values in the table show the volume V of water remaining in the tank (in gallons) after t minutes.

t (min)	5	10	15	20	25	30
V (gal)	694	444	250	111	28	0

(a) Find the average rates at which water flows from the tank (slopes of secant lines) for the time intervals $[10, 15]$ and $[15, 20]$.

(b) The slope of the tangent line at the point $(15, 250)$ represents the rate at which water is flowing from the tank after 15 minutes. Estimate this rate by averaging the slopes of the secant lines in part (a).

32. World Population Growth The table gives the world's population in the 20th century.

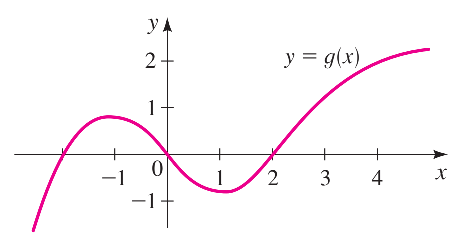
Year	Population (in millions)	Year	Population (in millions)
1900	1650	1960	3040
1910	1750	1970	3710
1920	1860	1980	4450
1930	2070	1990	5280
1940	2300	2000	6080
1950	2560		

Estimate the rate of population growth in 1920 and in 1980 by averaging the slopes of two secant lines.

Discovery • Discussion

33. Estimating Derivatives from a Graph For the function g whose graph is given, arrange the following numbers in increasing order and explain your reasoning.

$$0 \qquad g'(-2) \qquad g'(0) \qquad g'(2) \qquad g'(4)$$



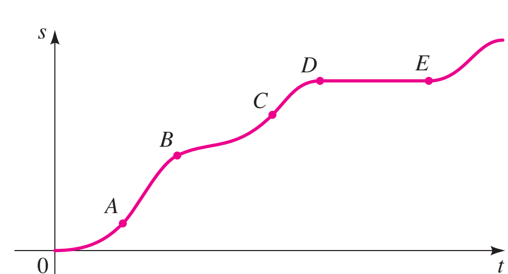
34. Estimating Velocities from a Graph The graph shows the position function of a car. Use the shape of the graph to explain your answers to the following questions.

(a) What was the initial velocity of the car?

(b) Was the car going faster at B or at C?

(c) Was the car slowing down or speeding up at A, B, and C?

(d) What happened between D and E?



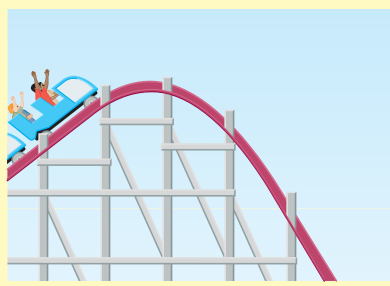
DISCOVERY PROJECT

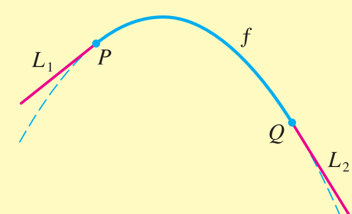
Designing a Roller Coaster

Suppose you are asked to design the first ascent and drop for a new roller coaster. By studying photographs of your favorite coasters, you decide to make the slope of the ascent 0.8 and the slope of the drop -1.6. You then connect these two straight stretches $y = L_1(x)$ and $y = L_2(x)$ with part of a parabola

$$y = f(x) = ax^2 + bx + c$$

where x and $f(x)$ are measured in feet. For the track to be smooth there can't be abrupt changes in direction, so you want the linear segments L_1 and L_2 to be tangent to the parabola at the transition points P and Q, as shown in the figure.





1. To simplify the equations, you decide to place the origin at P. As a consequence, what is the value of c?

2. Suppose the horizontal distance between P and Q is 100 ft. To ensure that the track is smooth at the transition points, what should the values of $f'(0)$ and $f'(100)$ be?

3. If $f(x) = ax^2 + bx + c$, show that $f'(x) = 2ax + b$.

4. Use the results of problems 2 and 3 to determine the values of a and b. That is, find a formula for $f(x)$.

5. Plot L_1, f, and L_2 to verify graphically that the transitions are smooth.

6. Find the difference in elevation between P and Q.

12.4 Limits at Infinity; Limits of Sequences

In this section we study a special kind of limit called a *limit at infinity*. We examine the limit of a function $f(x)$ as x becomes large. We also examine the limit of a sequence a_n as n becomes large. Limits of sequences will be used in Section 12.5 to help us find the area under the graph of a function.

Limits at Infinity

x	$f(x)$
0	−1.000000
±1	0.000000
±2	0.600000
±3	0.800000
±4	0.882353
±5	0.923077
±10	0.980198
±50	0.999200
±100	0.999800
±1000	0.999998

Let's investigate the behavior of the function f defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

as x becomes large. The table in the margin gives values of this function correct to six decimal places, and the graph of f has been drawn by a computer in Figure 1.

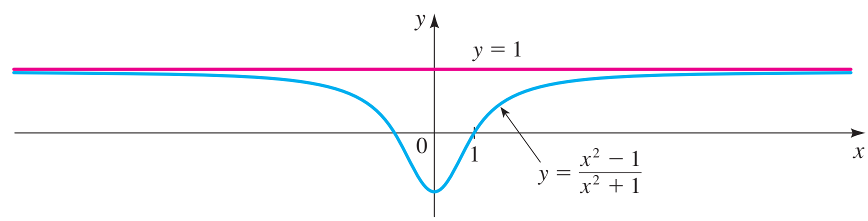


Figure 1

As x grows larger and larger, you can see that the values of $f(x)$ get closer and closer to 1. In fact, it seems that we can make the values of $f(x)$ as close as we like to 1 by taking x sufficiently large. This situation is expressed symbolically by writing

$$\lim_{x\to\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

In general, we use the notation

$$\lim_{x\to\infty} f(x) = L$$

to indicate that the values of $f(x)$ become closer and closer to L as x becomes larger and larger.

Limit at Infinity

Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

Another notation for $\lim_{x\to\infty} f(x) = L$ is

$$f(x) \to L \qquad \text{as} \qquad x \to \infty$$

Limits at infinity are also discussed in Section 3.6.

The symbol ∞ does not represent a number. Nevertheless, we often read the expression $\lim_{x\to\infty} f(x) = L$ as

"the limit of $f(x)$, as x approaches infinity, is L"

or "the limit of $f(x)$, as x becomes infinite, is L"

or "the limit of $f(x)$, as x increases without bound, is L"

Geometric illustrations are shown in Figure 2. Notice that there are many ways for the graph of f to approach the line $y = L$ (which is called a *horizontal asymptote*) as we look to the far right.

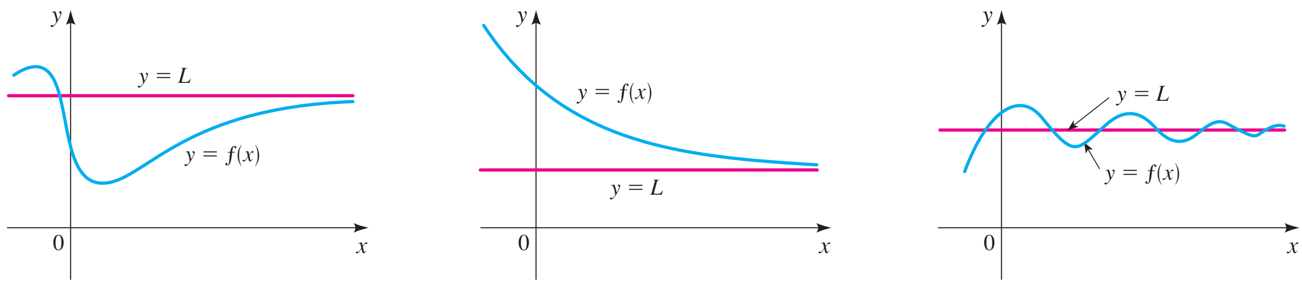


Figure 2
Examples illustrating $\lim_{x\to\infty} f(x) = L$

Referring back to Figure 1, we see that for numerically large negative values of x, the values of $f(x)$ are close to 1. By letting x decrease through negative values without bound, we can make $f(x)$ as close as we like to 1. This is expressed by writing

$$\lim_{x\to-\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

The general definition is as follows.

Limit at Negative Infinity

Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

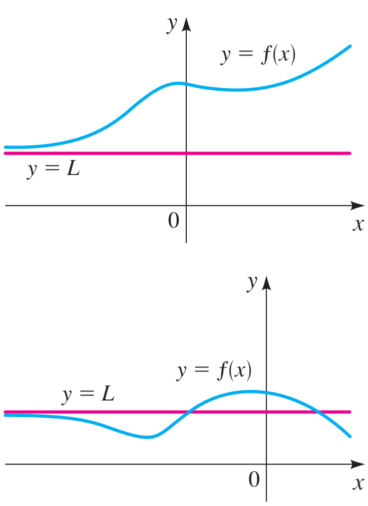


Figure 3
Examples illustrating $\lim_{x\to-\infty} f(x) = L$

Again, the symbol $-\infty$ does not represent a number, but the expression $\lim_{x\to-\infty} f(x) = L$ is often read as

"the limit of $f(x)$, as x approaches negative infinity, is L"

The definition is illustrated in Figure 3. Notice that the graph approaches the line $y = L$ as we look to the far left.

Horizontal Asymptote

The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L$$

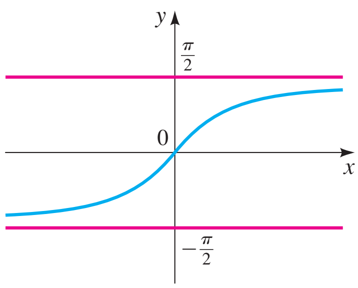


Figure 4
$y = \tan^{-1} x$

For instance, the curve illustrated in Figure 1 has the line $y = 1$ as a horizontal asymptote because

$$\lim_{x\to\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

As we discovered in Section 7.4, an example of a curve with two horizontal asymptotes is $y = \tan^{-1} x$ (see Figure 4). In fact,

$$\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2} \qquad \text{and} \qquad \lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$$

so both of the lines $y = -\pi/2$ and $y = \pi/2$ are horizontal asymptotes. (This follows from the fact that the lines $x = \pm\pi/2$ are vertical asymptotes of the graph of tan.)

Example 1 Limits at Infinity

Find $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$.

We first investigated horizontal asymptotes and limits at infinity for rational functions in Section 3.6.

Solution Observe that when x is large, $1/x$ is small. For instance,

$$\frac{1}{100} = 0.01 \qquad \frac{1}{10{,}000} = 0.0001 \qquad \frac{1}{1{,}000{,}000} = 0.000001$$

In fact, by taking x large enough, we can make $1/x$ as close to 0 as we please. Therefore

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

Similar reasoning shows that when x is large negative, $1/x$ is small negative, so we also have

$$\lim_{x\to-\infty} \frac{1}{x} = 0$$

It follows that the line $y = 0$ (the x-axis) is a horizontal asymptote of the curve $y = 1/x$. (This is a hyperbola; see Figure 5.) ■

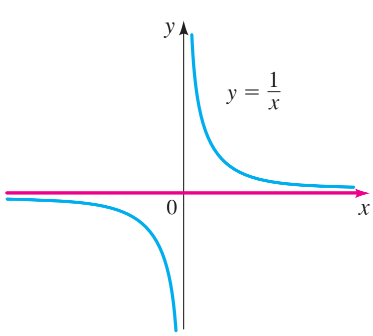


Figure 5
$\lim_{x\to\infty} \frac{1}{x} = 0, \ \lim_{x\to-\infty} \frac{1}{x} = 0$

The Limit Laws that we studied in Section 12.2 also hold for limits at infinity. In particular, if we combine Law 6 (Limit of a Power) with the results of Example 1, we obtain the following important rule for calculating limits.

> If k is any positive integer, then
>
> $$\lim_{x\to\infty} \frac{1}{x^k} = 0 \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{1}{x^k} = 0$$

Example 2 Finding a Limit at Infinity

Evaluate $\lim_{x\to\infty} \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$.

Solution To evaluate the limit at infinity of a rational function, we first divide both the numerator and denominator by the highest power of x that occurs in the denominator. (We may assume that $x \neq 0$ since we are interested only in large values of x.) In this case, the highest power of x in the denominator is x^2, so we have

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}}$$ Divide numerator and denominator by x^2

$$= \frac{\lim_{x\to\infty}\left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{\lim_{x\to\infty}\left(5 + \frac{4}{x} + \frac{1}{x^2}\right)}$$ Limit of a Quotient

$$= \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty} \frac{1}{x} - 2\lim_{x\to\infty} \frac{1}{x^2}}{\lim_{x\to\infty} 5 + 4\lim_{x\to\infty} \frac{1}{x} + \lim_{x\to\infty} \frac{1}{x^2}}$$ Limits of Sums and Differences

$$= \frac{3 - 0 - 0}{5 + 0 + 0} = \frac{3}{5}$$ Let $x \to \infty$

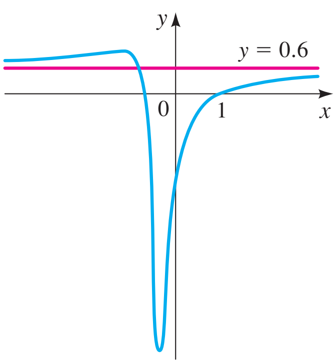


Figure 6

A similar calculation shows that the limit as $x \to -\infty$ is also $\frac{3}{5}$. Figure 6 illustrates the results of these calculations by showing how the graph of the given rational function approaches the horizontal asymptote $y = \frac{3}{5}$. ■

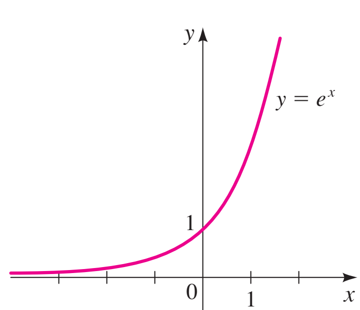
Example 3 A Limit at Negative Infinity

Use numerical and graphical methods to find $\lim_{x\to-\infty} e^x$.

Solution From the graph of the natural exponential function $y = e^x$ in Figure 7 and the corresponding table of values, we see that

$$\lim_{x\to-\infty} e^x = 0$$

It follows that the line $y = 0$ (the x-axis) is a horizontal asymptote.



x	e^x
0	1.00000
−1	0.36788
−2	0.13534
−3	0.04979
−5	0.00674
−8	0.00034
−10	0.00005

Figure 7

■

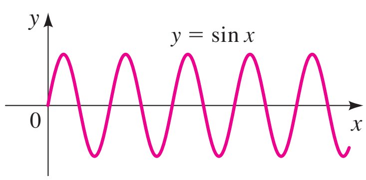


Figure 8

Example 4 A Function with No Limit at Infinity

Evaluate $\lim_{x\to\infty} \sin x$.

Solution From the graph in Figure 8 and the periodic nature of the sine function, we see that, as x increases, the values of $\sin x$ oscillate between 1 and -1 infinitely often and so they don't approach any definite number. Therefore, $\lim_{x\to\infty} \sin x$ does not exist. ■

Limits of Sequences

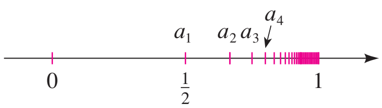


Figure 9

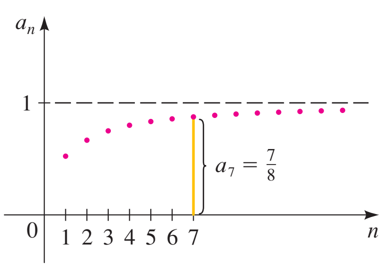


Figure 10

In Section 11.1 we introduced the idea of a sequence of numbers $a_1, a_2, a_3, \ldots$. Here we are interested in their behavior as n becomes large. For instance, the sequence defined by

$$a_n = \frac{n}{n+1}$$

is pictured in Figure 9 by plotting its terms on a number line and in Figure 10 by plotting its graph. From Figure 9 or 10 it appears that the terms of the sequence $a_n = n/(n+1)$ are approaching 1 as n becomes large. We indicate this by writing

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

Definition of the Limit of a Sequence

A sequence $a_1, a_2, a_3, \ldots$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if the nth term a_n of the sequence can be made arbitrarily close to L by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

This definition is illustrated by Figure 11.

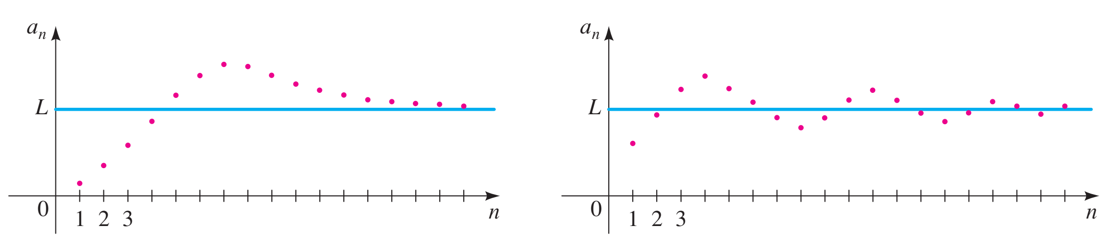


Figure 11
Graphs of two sequences with $\lim_{n\to\infty} a_n = L$

If we compare the definitions of $\lim_{n\to\infty} a_n = L$ and $\lim_{x\to\infty} f(x) = L$, we see that the only difference is that n is required to be an integer. Thus, the following is true.

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

In particular, since we know that $\lim_{x\to\infty}(1/x^k) = 0$ when k is a positive integer, we have

$$\lim_{n\to\infty} \frac{1}{n^k} = 0 \qquad \text{if } k \text{ is a positive integer}$$

Note that the Limit Laws given in Section 12.2 also hold for limits of sequences.

Example 5 Finding the Limit of a Sequence

Find $\lim_{n\to\infty} \dfrac{n}{n+1}$.

Solution The method is similar to the one we used in Example 2: Divide the numerator and denominator by the highest power of n and then use the Limit Laws.

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\dfrac{1}{n}} \qquad \text{Divide numerator and denominator by } n$$

$$= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \dfrac{1}{n}} \qquad \text{Limits of a Quotient and a Sum}$$

$$= \frac{1}{1+0} = 1 \qquad \text{Let } n \to \infty$$

This result shows that the guess we made earlier from Figures 9 and 10 was correct.

Therefore, the sequence $a_n = n/(n+1)$ is convergent. ■

Example 6 A Sequence That Diverges

Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

Solution If we write out the terms of the sequence, we obtain

$$-1, 1, -1, 1, -1, 1, -1, \ldots$$

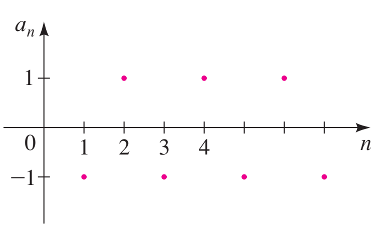


Figure 12

The graph of this sequence is shown in Figure 12. Since the terms oscillate between 1 and -1 infinitely often, a_n does not approach any number. Thus, $\lim_{n\to\infty}(-1)^n$ does not exist; that is, the sequence $a_n = (-1)^n$ is divergent. ■

Example 7 Finding the Limit of a Sequence

Find the limit of the sequence given by

$$a_n = \frac{15}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

Solution Before calculating the limit, let's first simplify the expression for a_n. Because $n^3 = n \cdot n \cdot n$, we place a factor of n beneath each factor in the numerator that contains an n:

$$a_n = \frac{15}{6} \cdot \frac{n}{n} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n} = \frac{5}{2} \cdot 1 \cdot \left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

Now we can compute the limit:

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{5}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) && \text{Definition of } a_n \\ &= \frac{5}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\lim_{n\to\infty}\left(2+\frac{1}{n}\right) && \text{Limit of a Product} \\ &= \frac{5}{2}(1)(2) = 5 && \text{Let } n\to\infty\end{aligned}$$

12.4 Exercises

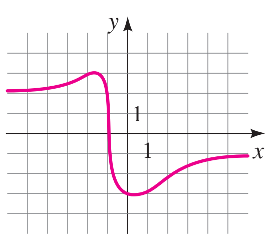
1–2 ■ **(a)** Use the graph of f to find the following limits.

(i) $\lim_{x\to\infty} f(x)$

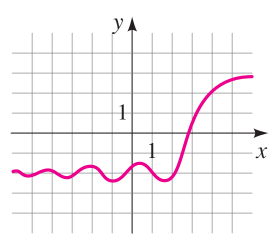
(ii) $\lim_{x\to\infty} f(x)$

(b) State the equations of the horizontal asymptotes.

1.



2.



3–14 ■ Find the limit.

3. $\lim_{x\to\infty} \frac{6}{x}$

4. $\lim_{x\to\infty} \frac{3}{x^4}$

5. $\lim_{x\to\infty} \frac{2x+1}{5x-1}$

6. $\lim_{x\to\infty} \frac{2-3x}{4x+5}$

7. $\lim_{x\to-\infty} \frac{4x^2+1}{2+3x^2}$

8. $\lim_{x\to-\infty} \frac{x^2+2}{x^3+x+1}$

9. $\lim_{t\to\infty} \frac{8t^3+t}{(2t-1)(2t^2+1)}$

10. $\lim_{r\to\infty} \frac{4r^3-r^2}{(r+1)^3}$

11. $\lim_{x\to\infty} \frac{x^4}{1-x^2+x^3}$

12. $\lim_{t\to\infty} \left(\frac{1}{t} - \frac{2t}{t-1}\right)$

13. $\lim_{x\to-\infty} \left(\frac{x-1}{x+1} + 6\right)$

14. $\lim_{x\to\infty} \cos x$

15–18 ■ Use a table of values to estimate the limit. Then use a graphing device to confirm your result graphically.

15. $\lim_{x\to-\infty} \frac{\sqrt{x^2+4x}}{4x+1}$

16. $\lim_{x\to\infty} \left(\sqrt{9x^2+x} - 3x\right)$

17. $\lim_{x\to\infty} \frac{x^5}{e^x}$

18. $\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^{3x}$

19–30 ■ If the sequence is convergent, find its limit. If it is divergent, explain why.

19. $a_n = \frac{1+n}{n+n^2}$

20. $a_n = \frac{5n}{n+5}$

21. $a_n = \frac{n^2}{n+1}$

22. $a_n = \frac{n-1}{n^3+1}$

23. $a_n = \frac{1}{3^n}$

24. $a_n = \frac{(-1)^n}{n}$

25. $a_n = \sin(n\pi/2)$

26. $a_n = \cos n\pi$

27. $a_n = \frac{3}{n^2}\left[\frac{n(n+1)}{2}\right]$

28. $a_n = \frac{5}{n}\left(n + \frac{4}{n}\left[\frac{n(n+1)}{2}\right]\right)$

29. $a_n = \frac{24}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$

30. $a_n = \frac{12}{n^4}\left[\frac{n(n+1)}{2}\right]^2$

Applications

31. Salt Concentration

(a) A tank contains 5000 L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min. Show that the concentration of salt after t minutes (in grams per liter) is

$$C(t) = \frac{30t}{200+t}$$

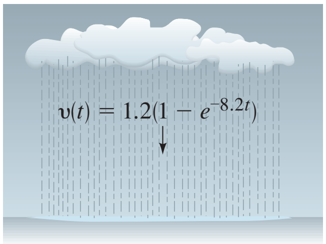
(b) What happens to the concentration as $t\to\infty$?

32. Velocity of a Raindrop The downward velocity of a falling raindrop at time t is modeled by the function

$$v(t) = 1.2(1 - e^{-8.2t})$$

(a) Find the terminal velocity of the raindrop by evaluating $\lim_{t\to\infty} v(t)$. (Use the result of Example 3.)

(b) Graph $v(t)$, and use the graph to estimate how long it takes for the velocity of the raindrop to reach 99% of its terminal velocity.



Discovery • Discussion

33. The Limit of a Recursive Sequence

(a) A sequence is defined recursively by $a_1 = 0$ and

$$a_{n+1} = \sqrt{2 + a_n}$$

Find the first ten terms of this sequence correct to eight decimal places. Does this sequence appear to be convergent? If so, guess the value of the limit.

(b) Assuming the sequence in part (a) is convergent, let $\lim_{n\to\infty} a_n = L$. Explain why $\lim_{n\to\infty} a_{n+1} = L$ also, and therefore

$$L = \sqrt{2 + L}$$

Solve this equation to find the exact value of L.

12.5 Areas

We have seen that limits are needed to compute the slope of a tangent line or an instantaneous rate of change. Here we will see that they are also needed to find the area of a region with a curved boundary. The problem of finding such areas has consequences far beyond simply finding area. (See *Focus on Modeling*, page 929.)

The Area Problem

One of the central problems in calculus is the *area problem*: Find the area of the region S that lies under the curve $y = f(x)$ from a to b. This means that S, illustrated in Figure 1, is bounded by the graph of a function f (where $f(x) \geq 0$), the vertical lines $x = a$ and $x = b$, and the x-axis.

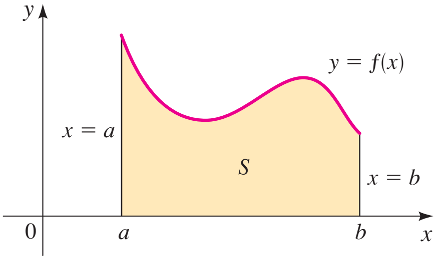


Figure 1

In trying to solve the area problem, we have to ask ourselves: What is the meaning of the word *area*? This question is easy to answer for regions with straight sides.

For a rectangle, the area is defined as the product of the length and the width. The area of a triangle is half the base times the height. The area of a polygon is found by dividing it into triangles (as in Figure 2) and adding the areas of the triangles.

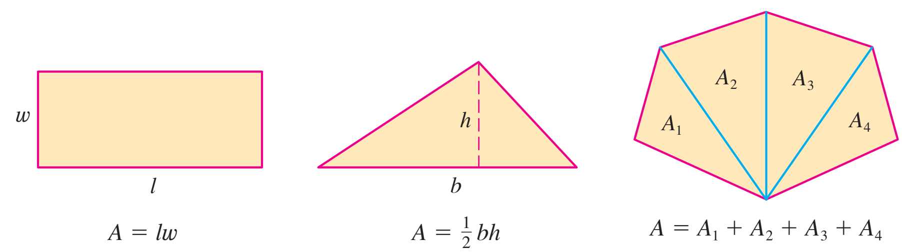


Figure 2 $A = lw$ $A = \frac{1}{2}bh$ $A = A_1 + A_2 + A_3 + A_4$

However, it is not so easy to find the area of a region with curved sides. We all have an intuitive idea of what the area of a region is. But part of the area problem is to make this intuitive idea precise by giving an exact definition of area.

Recall that in defining a tangent we first approximated the slope of the tangent line by slopes of secant lines and then we took the limit of these approximations. We pursue a similar idea for areas. We first approximate the region S by rectangles, and then we take the limit of the areas of these rectangles as we increase the number of rectangles. The following example illustrates the procedure.

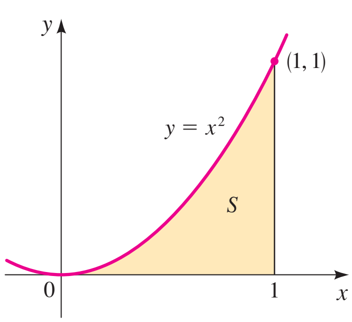


Figure 3

Example 1 Estimating an Area Using Rectangles

Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1 (the parabolic region S illustrated in Figure 3).

Solution We first notice that the area of S must be somewhere between 0 and 1 because S is contained in a square with side length 1, but we can certainly do better than that. Suppose we divide S into four strips S_1, S_2, S_3, and S_4 by drawing the vertical lines $x = \frac{1}{4}$, $x = \frac{1}{2}$, and $x = \frac{3}{4}$ as in Figure 4(a). We can approximate each strip by a rectangle whose base is the same as the strip and whose height is the same as the right edge of the strip (see Figure 4(b)). In other words, the heights of these rectangles are the values of the function $f(x) = x^2$ at the right endpoints of the subintervals $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{1}{2}]$, $[\frac{1}{2}, \frac{3}{4}]$, and $[\frac{3}{4}, 1]$.

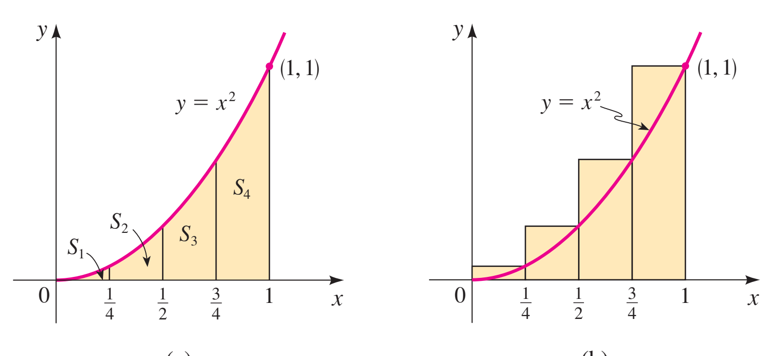


Figure 4

Each rectangle has width $\frac{1}{4}$ and the heights are $(\frac{1}{4})^2$, $(\frac{1}{2})^2$, $(\frac{3}{4})^2$, and 1^2. If we let R_4 be the sum of the areas of these approximating rectangles, we get

$$R_4 = \tfrac{1}{4}\cdot(\tfrac{1}{4})^2 + \tfrac{1}{4}\cdot(\tfrac{1}{2})^2 + \tfrac{1}{4}\cdot(\tfrac{3}{4})^2 + \tfrac{1}{4}\cdot 1^2 = \tfrac{15}{32} = 0.46875$$

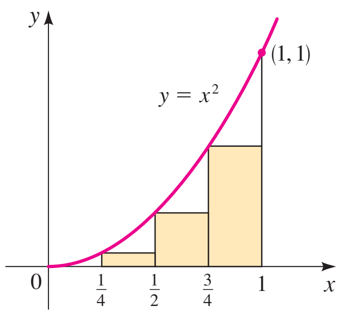


Figure 5

From Figure 4(b) we see that the area A of S is less than R_4, so

$$A < 0.46875$$

Instead of using the rectangles in Figure 4(b), we could use the smaller rectangles in Figure 5 whose heights are the values of f at the left endpoints of the subintervals. (The leftmost rectangle has collapsed because its height is 0.) The sum of the areas of these approximating rectangles is

$$L_4 = \tfrac{1}{4}\cdot 0^2 + \tfrac{1}{4}\cdot(\tfrac{1}{4})^2 + \tfrac{1}{4}\cdot(\tfrac{1}{2})^2 + \tfrac{1}{4}\cdot(\tfrac{3}{4})^2 = \tfrac{7}{32} = 0.21875$$

We see that the area of S is larger than L_4, so we have lower and upper estimates for A:

$$0.21875 < A < 0.46875$$

We can repeat this procedure with a larger number of strips. Figure 6 shows what happens when we divide the region S into eight strips of equal width. By computing the sum of the areas of the smaller rectangles (L_8) and the sum of the areas of the larger rectangles (R_8), we obtain better lower and upper estimates for A:

$$0.2734375 < A < 0.3984375$$

So one possible answer to the question is to say that the true area of S lies somewhere between 0.2734375 and 0.3984375.

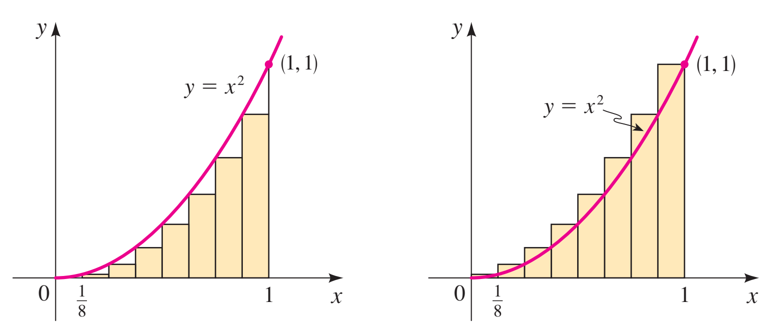


Figure 6

Approximating S with eight rectangles

(a) Using left endpoints (b) Using right endpoints

We could obtain better estimates by increasing the number of strips. The table in the margin shows the results of similar calculations (with a computer) using n rectangles whose heights are found with left endpoints (L_n) or right endpoints (R_n). In particular, we see by using 50 strips that the area lies between 0.3234 and 0.3434. With 1000 strips we narrow it down even more: A lies between 0.3328335 and 0.3338335. A good estimate is obtained by averaging these numbers: $A \approx 0.3333335$. ■

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

From the values in the table it looks as if R_n is approaching $\frac{1}{3}$ as n increases. We confirm this in the next example.

Example 2 The Limit of Approximating Sums

For the region S in Example 1, show that the sum of the areas of the upper approximating rectangles approaches $\frac{1}{3}$, that is,

$$\lim_{n\to\infty} R_n = \tfrac{1}{3}$$

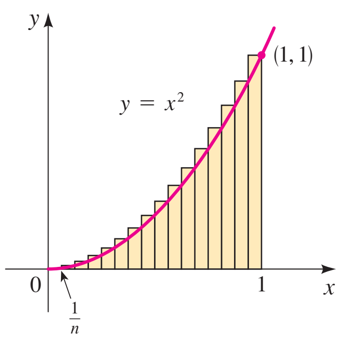


Figure 7

Solution R_n is the sum of the areas of the n rectangles shown in Figure 7. Each rectangle has width $1/n$, and the heights are the values of the function $f(x) = x^2$ at the points $1/n, 2/n, 3/n, \ldots, n/n$. That is, the heights are $(1/n)^2, (2/n)^2, (3/n)^2, \ldots, (n/n)^2$. Thus

$$\begin{aligned} R_n &= \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2 \\ &= \frac{1}{n}\cdot\frac{1}{n^2}(1^2 + 2^2 + 3^2 + \cdots + n^2) \\ &= \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2) \end{aligned}$$

Here we need the formula for the sum of the squares of the first n positive integers:

This formula was discussed in Section 11.5.

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Putting the preceding formula into our expression for R_n, we get

$$R_n = \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

Thus, we have

$$\begin{aligned} \lim_{n\to\infty} R_n &= \lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2} \\ &= \lim_{n\to\infty}\frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) \\ &= \lim_{n\to\infty}\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \\ &= \tfrac{1}{6}\cdot 1\cdot 2 = \tfrac{1}{3} \end{aligned}$$

■

It can be shown that the lower approximating sums also approach $\frac{1}{3}$, that is,

$$\lim_{n\to\infty} L_n = \tfrac{1}{3}$$

From Figures 8 and 9 it appears that, as n increases, both R_n and L_n become better and better approximations to the area of S. Therefore, we *define* the area A to be the limit of the sums of the areas of the approximating rectangles, that is,

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \tfrac{1}{3}$$

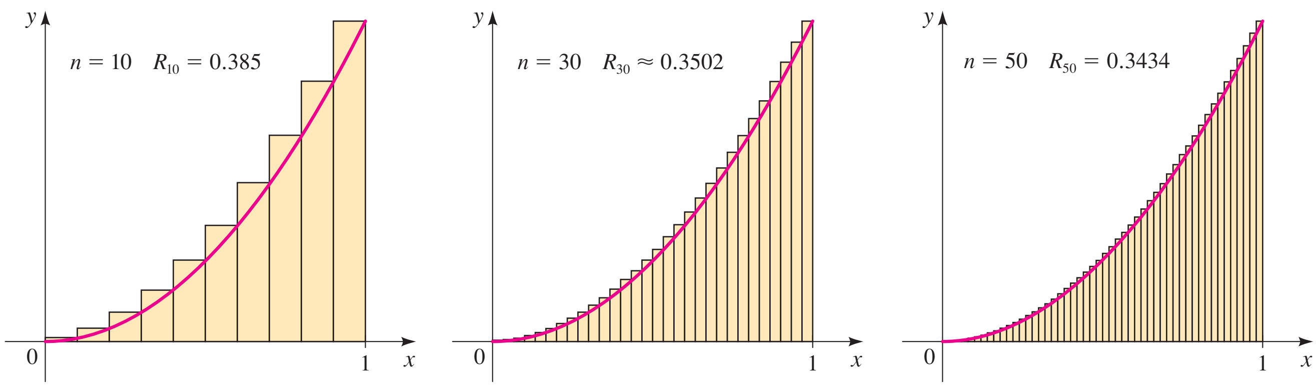


Figure 8

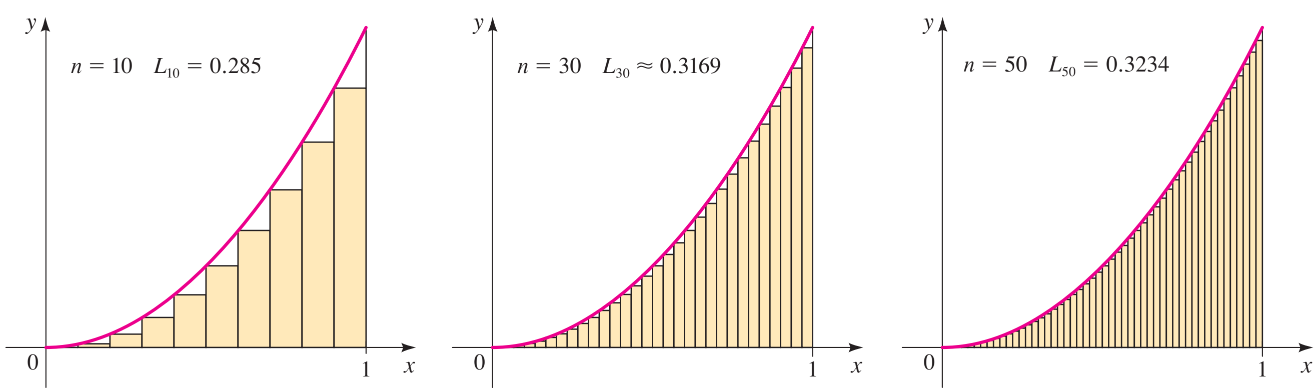


Figure 9

Definition of Area

Let's apply the idea of Examples 1 and 2 to the more general region S of Figure 1. We start by subdividing S into n strips $S_1, S_2, \ldots, S_n$ of equal width as in Figure 10.

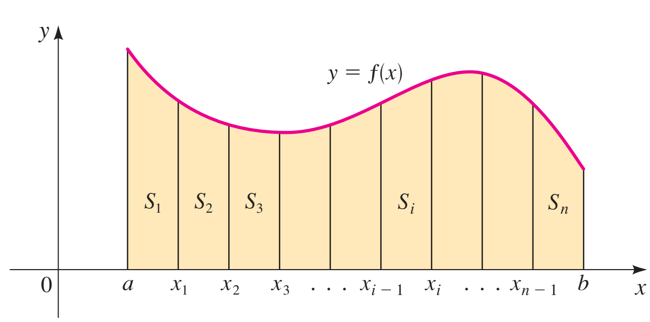


Figure 10

The width of the interval $[a, b]$ is $b - a$, so the width of each of the n strips is

$$\Delta x = \frac{b - a}{n}$$

These strips divide the interval $[a, b]$ into n subintervals

$$[x_0, x_1], \quad [x_1, x_2], \quad [x_2, x_3], \quad \dots, \quad [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$. The right endpoints of the subintervals are

$$x_1 = a + \Delta x, \quad x_2 = a + 2\,\Delta x, \quad x_3 = a + 3\,\Delta x, \quad \dots, \quad x_k = a + k\,\Delta x, \quad \dots$$

Let's approximate the kth strip S_k by a rectangle with width Δx and height $f(x_k)$, which is the value of f at the right endpoint (see Figure 11). Then the area of the kth rectangle is $f(x_k)\Delta x$. What we think of intuitively as the area of S is approximated by the sum of the areas of these rectangles, which is

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

Figure 12 shows this approximation for $n = 2, 4, 8$, and 12.

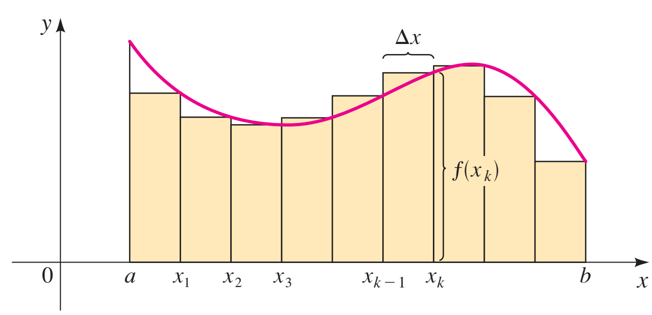


Figure 11

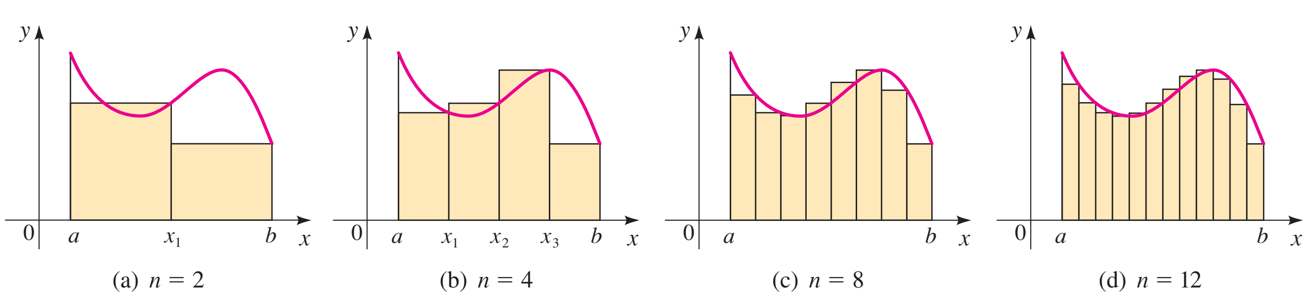


Figure 12

Notice that this approximation appears to become better and better as the number of strips increases, that is, as $n \to \infty$. Therefore, we define the area A of the region S in the following way.

Definition of Area

The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

Using sigma notation, we write this as follows:

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

In using this formula for area, remember that Δx is the width of an approximating rectangle, x_k is the right endpoint of the kth rectangle, and $f(x_k)$ is its height. So

Width: $\Delta x = \dfrac{b-a}{n}$

Right endpoint: $x_k = a + k\,\Delta x$

Height: $f(x_k) = f(a + k\,\Delta x)$

When working with sums, we will need the following properties from Section 11.1:

$$\sum_{k=1}^{n}(a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k \qquad \sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$$

We will also need the following formulas for the sums of the powers of the first n natural numbers from Section 11.5.

$$\sum_{k=1}^{n} c = nc \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

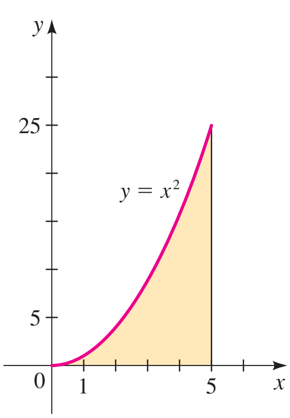


Figure 13

Example 3 Finding the Area under a Curve

Find the area of the region that lies under the parabola $y = x^2$, $0 \le x \le 5$.

Solution The region is graphed in Figure 13. To find the area, we first find the dimensions of the approximating rectangles at the nth stage.

Width: $\Delta x = \dfrac{b-a}{n} = \dfrac{5-0}{n} = \dfrac{5}{n}$

Right endpoint: $x_k = a + k\,\Delta x = 0 + k\left(\dfrac{5}{n}\right) = \dfrac{5k}{n}$

Height: $f(x_k) = f\left(\dfrac{5k}{n}\right) = \left(\dfrac{5k}{n}\right)^2 = \dfrac{25k^2}{n^2}$

Now we substitute these values into the definition of area:

$$
\begin{aligned}
A &= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x && \text{Definition of area} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{25k^2}{n^2}\cdot\frac{5}{n} && f(x_k) = \frac{25k^2}{n^2},\ \Delta x = \frac{5}{n} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{125k^2}{n^3} && \text{Simplify} \\
&= \lim_{n\to\infty} \frac{125}{n^3}\sum_{k=1}^{n} k^2 && \text{Factor } \frac{125}{n^3} \\
&= \lim_{n\to\infty} \frac{125}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} && \text{Sum of squares formula} \\
&= \lim_{n\to\infty} \frac{125(2n^2+3n+1)}{6n^2} && \text{Cancel } n \text{ and expand numerator} \\
&= \lim_{n\to\infty} \frac{125}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right) && \text{Divide numerator and denominator by } n^2 \\
&= \frac{125}{6}(2+0+0) = \frac{125}{3} && \text{Let } n\to\infty
\end{aligned}
$$

We can also calculate the limit by writing

$$
\frac{125}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{125}{6}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)
$$

as in Example 2.

Thus, the area of the region is $\frac{125}{3} \approx 41.7$. ■

Example 4 Finding the Area under a Curve

Find the area of the region that lies under the parabola $y = 4x - x^2$, $1 \le x \le 3$.

Solution We start by finding the dimensions of the approximating rectangles at the nth stage.

$$
\text{Width:} \qquad \Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}
$$

$$
\text{Right endpoint:} \qquad x_k = a + k\Delta x = 1 + k\left(\frac{2}{n}\right) = 1 + \frac{2k}{n}
$$

$$
\begin{aligned}
\text{Height:} \qquad f(x_k) &= f\left(1+\frac{2k}{n}\right) = 4\left(1+\frac{2k}{n}\right) - \left(1+\frac{2k}{n}\right)^2 \\
&= 4 + \frac{8k}{n} - 1 - \frac{4k}{n} - \frac{4k^2}{n^2} \\
&= 3 + \frac{4k}{n} - \frac{4k^2}{n^2}
\end{aligned}
$$

Figure 14 shows the region whose area is computed in Example 4.

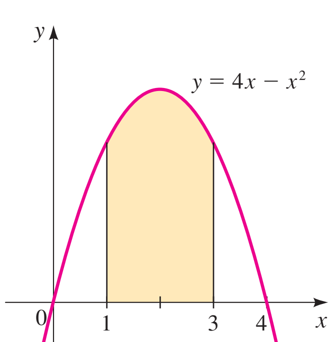


Figure 14

Thus, according to the definition of area, we get

$$
\begin{aligned}
A &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left(3+\frac{4k}{n}-\frac{4k^2}{n^2}\right)\left(\frac{2}{n}\right) \\
&= \lim_{n\to\infty}\left(\sum_{k=1}^{n} 3 + \frac{4}{n}\sum_{k=1}^{n} k - \frac{4}{n^2}\sum_{k=1}^{n} k^2\right)\left(\frac{2}{n}\right)
\end{aligned}
$$

$$= \lim_{n\to\infty}\left(\frac{2}{n}\sum_{k=1}^{n}3 + \frac{8}{n^2}\sum_{k=1}^{n}k - \frac{8}{n^3}\sum_{k=1}^{n}k^2\right)$$

$$= \lim_{n\to\infty}\left(\frac{2}{n}(3n) + \frac{8}{n^2}\left[\frac{n(n+1)}{2}\right] - \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]\right)$$

$$= \lim_{n\to\infty}\left(6 + 4\cdot\frac{n}{n}\cdot\frac{n+1}{n} - \frac{4}{3}\cdot\frac{n}{n}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}\right)$$

$$= \lim_{n\to\infty}\left[6 + 4\left(1+\frac{1}{n}\right) - \frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]$$

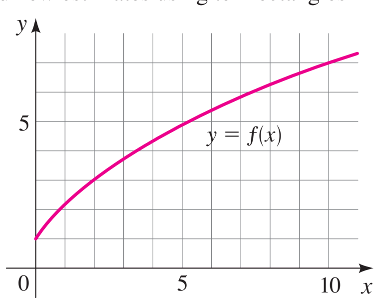
$$= 6 + 4\cdot 1 - \frac{4}{3}\cdot 1\cdot 2 = \frac{22}{3}$$

■

12.5 Exercises

1. **(a)** By reading values from the given graph of f, use five rectangles to find a lower estimate and an upper estimate for the area under the given graph of f from $x = 0$ to $x = 10$. In each case, sketch the rectangles that you use.

(b) Find new estimates using ten rectangles in each case.



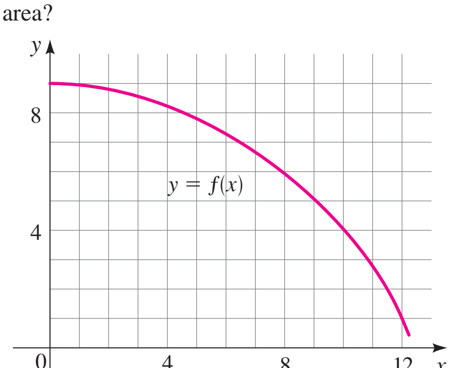
2. **(a)** Use six rectangles to find estimates of each type for the area under the given graph of f from $x = 0$ to $x = 12$.

(i) L_6 (using left endpoints)

(ii) R_6 (using right endpoints)

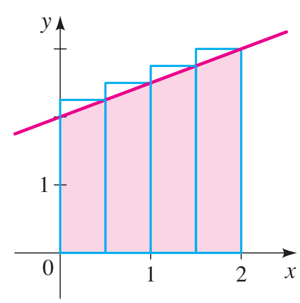
(b) Is L_6 an underestimate or an overestimate of the true area?

(c) Is R_6 an underestimate or an overestimate of the true area?

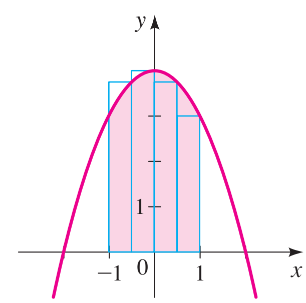


3–6 ■ Approximate the area of the shaded region under the graph of the given function by using the indicated rectangles. (The rectangles have equal width.)

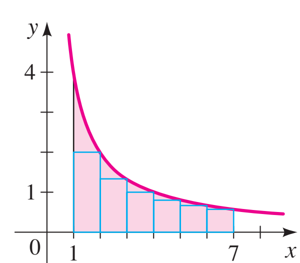
3. $f(x) = \frac{1}{2}x + 2$



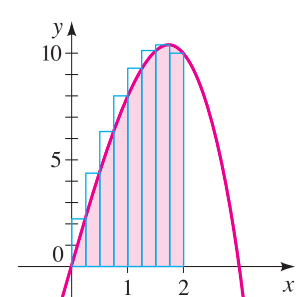
4. $f(x) = 4 - x^2$



5. $f(x) = \dfrac{4}{x}$



6. $f(x) = 9x - x^3$



7. **(a)** Estimate the area under the graph of $f(x) = 1/x$ from $x = 1$ to $x = 5$ using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?

(b) Repeat part (a) using left endpoints.

8. **(a)** Estimate the area under the graph of $f(x) = 25 - x^2$ from $x = 0$ to $x = 5$ using five approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?

(b) Repeat part (a) using left endpoints.

9. **(a)** Estimate the area under the graph of $f(x) = 1 + x^2$ from $x = -1$ to $x = 2$ using three rectangles and right endpoints. Then improve your estimate by using six rectangles. Sketch the curve and the approximating rectangles.

(b) Repeat part (a) using left endpoints.

10. **(a)** Estimate the area under the graph of $f(x) = e^{-x}$, $0 \le x \le 4$, using four approximating rectangles and taking the sample points to be

(i) right endpoints

(ii) left endpoints

In each case, sketch the curve and the rectangles.

(b) Improve your estimates in part (a) by using eight rectangles.

11–12 ■ Use the definition of area as a limit to find the area of the region that lies under the curve. Check your answer by sketching the region and using geometry.

11. $y = 3x, \quad 0 \le x \le 5$

12. $y = 2x + 1, \quad 1 \le x \le 3$

13–18 ■ Find the area of the region that lies under the graph of f over the given interval.

13. $f(x) = 3x^2, \quad 0 \le x \le 2$

14. $f(x) = x + x^2, \quad 0 \le x \le 1$

15. $f(x) = x^3 + 2, \quad 0 \le x \le 5$

16. $f(x) = 4x^3, \quad 2 \le x \le 5$

17. $f(x) = x + 6x^2, \quad 1 \le x \le 4$

18. $f(x) = 20 - 2x^2, \quad 2 \le x \le 3$

Discovery • Discussion

19. Approximating Area with a Calculator When we approximate areas using rectangles as in Example 1, then the more rectangles we use the more accurate the answer. The following TI-83 program finds the approximate area under the graph of f on the interval $[a, b]$ using n rectangles. To use the program, first store the function f in Y_1. The program prompts you to enter N, the number of rectangles, and A and B, the endpoints of the interval.

(a) Approximate the area under the graph of $f(x) = x^5 + 2x + 3$ on $[1, 3]$ using 10, 20, and 100 rectangles.

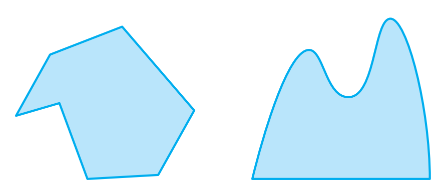
(b) Approximate the area under the graph of f on the given interval using 100 rectangles.

(i) $f(x) = \sin x, \quad$ on $[0, \pi]$

(ii) $f(x) = e^{-x^2}, \quad$ on $[-1, 1]$

```
PROGRAM:AREA
:Prompt N
:Prompt A
:Prompt B
:(B-A)/N→D
:0→S
:A→X
:For (K,1,N)
:X+D→X
:S+Y1→S
:End
:D*S→S
:Disp "AREA IS"
:Disp S
```

20. Regions with Straight Versus Curved Boundaries Write a short essay that explains how you would find the area of a polygon, that is, a region bounded by straight line segments. Then explain how you would find the area of a region whose boundary is curved, as we did in this section. What is the fundamental difference between these two processes?



12 Review

Concept Check

1. Explain in your own words what is meant by the equation

$$\lim_{x \to 2} f(x) = 5$$

Is it possible for this statement to be true and yet $f(2) = 3$? Explain.

2. Explain what it means to say that

$$\lim_{x \to 1^-} f(x) = 3 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = 7$$

In this situation is it possible that $\lim_{x \to 1} f(x)$ exists? Explain.

3. Describe several ways in which a limit can fail to exist. Illustrate with sketches.

4. State the following Limit Laws.

(a) Sum Law
(b) Difference Law
(c) Constant Multiple Law
(d) Product Law
(e) Quotient Law
(f) Power Law
(g) Root Law

5. Write an expression for the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$.

6. Define the derivative $f'(a)$. Discuss two ways of interpreting this number.

7. If $y = f(x)$, write expressions for the following.

(a) The average rate of change of y with respect to x between the numbers a and x.
(b) The instantaneous rate of change of y with respect to x at $x = a$.

8. Explain the meaning of the equation

$$\lim_{x\to\infty} f(x) = 2$$

Draw sketches to illustrate the various possibilities.

9. (a) What does it mean to say that the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$? Draw curves to illustrate the various possibilities.

(b) Which of the following curves have horizontal asymptotes?

(i) $y = x^2$ (iv) $y = \tan^{-1} x$
(ii) $y = 1/x$ (v) $y = e^x$
(iii) $y = \sin x$ (vi) $y = \ln x$

10. (a) What is a convergent sequence?
(b) What does $\lim_{n\to\infty} a_n = 3$ mean?

11. Suppose S is the region that lies under the graph of $y = f(x)$, $a \le x \le b$.

(a) Explain how this area is approximated using rectangles.
(b) Write an expression for the area of S as a limit of sums.

Exercises

1–6 ■ Use a table of values to estimate the value of the limit. Then use a graphing device to confirm your result graphically.

1. $\displaystyle\lim_{x\to 2} \frac{x-2}{x^2 - 3x + 2}$

2. $\displaystyle\lim_{t\to -1} \frac{t+1}{t^3 - t}$

3. $\displaystyle\lim_{x\to 0} \frac{2^x - 1}{x}$

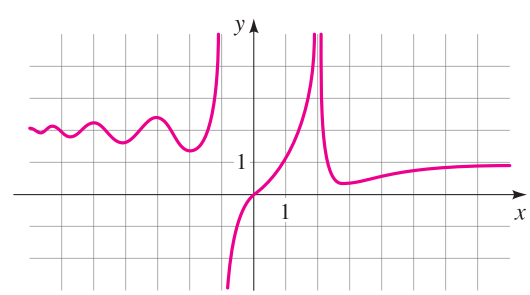
4. $\displaystyle\lim_{x\to 0} \frac{\sin 2x}{x}$

5. $\displaystyle\lim_{x\to 1^+} \ln\sqrt{x-1}$

6. $\displaystyle\lim_{x\to 0^-} \frac{\tan x}{|x|}$

7. The graph of f is shown in the figure. Find each limit or explain why it does not exist.

(a) $\lim_{x\to 2^+} f(x)$ **(b)** $\lim_{x\to -3^+} f(x)$
(c) $\lim_{x\to -3^-} f(x)$ **(d)** $\lim_{x\to -3} f(x)$
(e) $\lim_{x\to 4} f(x)$ **(f)** $\lim_{x\to\infty} f(x)$
(g) $\lim_{x\to -\infty} f(x)$ **(h)** $\lim_{x\to 0} f(x)$



8. Let

$$f(x) = \begin{cases} 2 & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x \le 2 \\ x + 2 & \text{if } x > 2 \end{cases}$$

Find each limit or explain why it does not exist.

(a) $\lim_{x\to -1^-} f(x)$ **(b)** $\lim_{x\to -1^+} f(x)$
(c) $\lim_{x\to -1} f(x)$ **(d)** $\lim_{x\to 2^-} f(x)$

(e) $\lim_{x \to 2^+} f(x)$ **(f)** $\lim_{x \to 2} f(x)$

(g) $\lim_{x \to 0} f(x)$ **(h)** $\lim_{x \to 3} (f(x))^2$

9–20 ■ Use the Limit Laws to evaluate the limit, if it exists.

9. $\lim_{x \to 2} \dfrac{x + 1}{x - 3}$

10. $\lim_{t \to 1} (t^3 - 3t + 6)$

11. $\lim_{x \to 3} \dfrac{x^2 + x - 12}{x - 3}$

12. $\lim_{x \to -2} \dfrac{x^2 - 4}{x^2 + x - 2}$

13. $\lim_{u \to 0} \dfrac{(u + 1)^2 - 1}{u}$

14. $\lim_{z \to 9} \dfrac{\sqrt{z} - 3}{z - 9}$

15. $\lim_{x \to 3^-} \dfrac{x - 3}{|x - 3|}$

16. $\lim_{x \to 0} \left(\dfrac{1}{x} + \dfrac{2}{x^2 - 2x} \right)$

17. $\lim_{x \to \infty} \dfrac{2x}{x - 4}$

18. $\lim_{x \to \infty} \dfrac{x^2 + 1}{x^4 - 3x + 6}$

19. $\lim_{x \to \infty} \cos^2 x$

20. $\lim_{t \to -\infty} \dfrac{t^4}{t^3 - 1}$

21–24 ■ Find the derivative of the function at the given number.

21. $f(x) = 3x - 5$, at 4

22. $g(x) = 2x^2 - 1$, at -1

23. $f(x) = \sqrt{x}$, at 16

24. $f(x) = \dfrac{x}{x + 1}$, at 1

25–28 ■ **(a)** Find $f'(a)$. **(b)** Find $f'(2)$ and $f'(-2)$.

25. $f(x) = 6 - 2x$

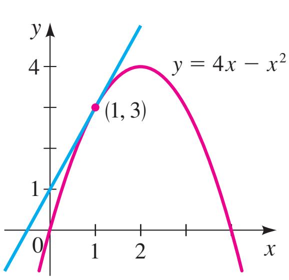
26. $f(x) = x^2 - 3x$

27. $f(x) = \sqrt{x + 6}$

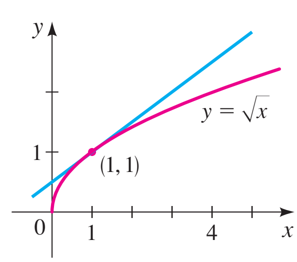
28. $f(x) = \dfrac{4}{x}$

29–30 ■ Find an equation of the tangent line shown in the figure.

29.



30.



31–34 ■ Find an equation of the line tangent to the graph of f at the given point.

31. $f(x) = 2x$, at $(3, 6)$

32. $f(x) = x^2 - 3$, at $(2, 1)$

33. $f(x) = \dfrac{1}{x}$, at $\left(2, \dfrac{1}{2}\right)$

34. $f(x) = \sqrt{x + 1}$, at $(3, 2)$

35. A stone is dropped from the roof of a building 640 ft above the ground. Its height (in feet) after t seconds is given by $h(t) = 640 - 16t^2$.

(a) Find the velocity of the stone when $t = 2$.

(b) Find the velocity of the stone when $t = a$.

(c) At what time t will the stone hit the ground?

(d) With what velocity will the stone hit the ground?

36. If a gas is confined in a fixed volume, then according to Boyle's Law the product of the pressure P and the temperature T is a constant. For a certain gas, $PT = 100$, where P is measured in lb/in^2 and T is measured in kelvins (K).

(a) Express P as a function of T.

(b) Find the instantaneous rate of change of P with respect to T when $T = 300$ K.

37–42 ■ If the sequence is convergent, find its limit. If it is divergent, explain why.

37. $a_n = \dfrac{n}{5n + 1}$

38. $a_n = \dfrac{n^3}{n^3 + 1}$

39. $a_n = \dfrac{n(n + 1)}{2n^2}$

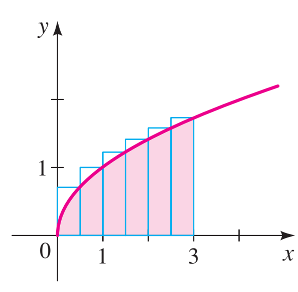
40. $a_n = \dfrac{n^3}{2n + 6}$

41. $a_n = \cos\left(\dfrac{n\pi}{2}\right)$

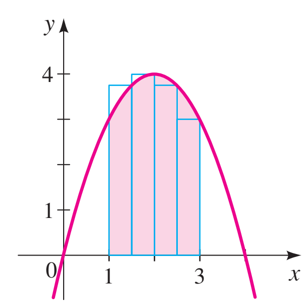
42. $a_n = \dfrac{10}{3^n}$

43–44 ■ Approximate the area of the shaded region under the graph of the given function by using the indicated rectangles. (The rectangles have equal width.)

43. $f(x) = \sqrt{x}$



44. $f(x) = 4x - x^2$



45–48 ■ Use the limit definition of area to find the area of the region that lies under the graph of f over the given interval.

45. $f(x) = 2x + 3$, $0 \le x \le 2$

46. $f(x) = x^2 + 1$, $0 \le x \le 3$

47. $f(x) = x^2 - x$, $1 \le x \le 2$

48. $f(x) = x^3$, $1 \le x \le 2$

12 Test

1. (a) Use a table of values to estimate the limit

$$\lim_{x \to 0} \frac{x}{\sin 2x}$$

(b) Use a graphing calculator to confirm your answer graphically.

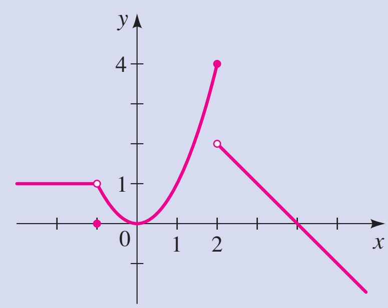
2. For the piecewise-defined function f whose graph is shown, find:

(a) $\lim_{x \to -1^-} f(x)$ (b) $\lim_{x \to -1^+} f(x)$ (c) $\lim_{x \to -1} f(x)$

(d) $\lim_{x \to 0^-} f(x)$ (e) $\lim_{x \to 0^+} f(x)$ (f) $\lim_{x \to 0} f(x)$

(g) $\lim_{x \to 2^-} f(x)$ (h) $\lim_{x \to 2^+} f(x)$ (i) $\lim_{x \to 2} f(x)$

$$f(x) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } x = -1 \\ x^2 & \text{if } -1 < x \le 2 \\ 4 - x & \text{if } 2 < x \end{cases}$$



3. Evaluate the limit, if it exists.

(a) $\lim_{x \to 2} \frac{x^2 + 2x - 8}{x - 2}$ (b) $\lim_{x \to 2} \frac{x^2 - 2x - 8}{x + 2}$ (c) $\lim_{x \to 2} \frac{1}{x - 2}$

(d) $\lim_{x \to 2} \frac{x - 2}{|x - 2|}$ (e) $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$ (f) $\lim_{x \to \infty} \frac{2x^2 - 4}{x^2 + x}$

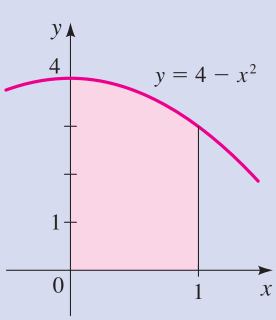
4. Let $f(x) = x^2 - 2x$. Find:

(a) $f'(x)$ (b) $f'(-1), f'(1), f'(2)$

5. Find the equation of the line tangent to the graph of $f(x) = \sqrt{x}$ at the point where $x = 9$.

6. Find the limit of the sequence.

(a) $a_n = \frac{n}{n^2 + 4}$ (b) $a_n = \sec n\pi$



7. The region sketched in the figure in the margin lies under the graph of $f(x) = 4 - x^2$, above the interval $0 \le x \le 1$.

(a) Approximate the area of the region with five rectangles, equally spaced along the x-axis, using right endpoints to determine the heights of the rectangles.

(b) Use the limit definition of area to find the exact value of the area of the region.

Focus on Modeling
Interpretations of Area

The area under the graph of a function is used to model many quantities in physics, economics, engineering, and other fields. That is why the area problem is so important. Here we will show how the concept of work (Section 8.5) is modeled by area. Several other applications are explored in the problems.

Recall that the work W done in moving an object is the product of the force F applied to the object and the distance d that the object moves:

$$W = Fd \qquad \text{work} = \text{force} \times \text{distance}$$

This formula is used if the force is *constant*. For example, suppose you are pushing a crate across a floor, moving along the positive x-axis from $x = a$ to $x = b$, and you apply a constant force $F = k$. The graph of F as a function of the distance x is shown in Figure 1(a). Notice that the work done is $W = Fd = k(b - a)$, which is the area under the graph of F (see Figure 1(b)).

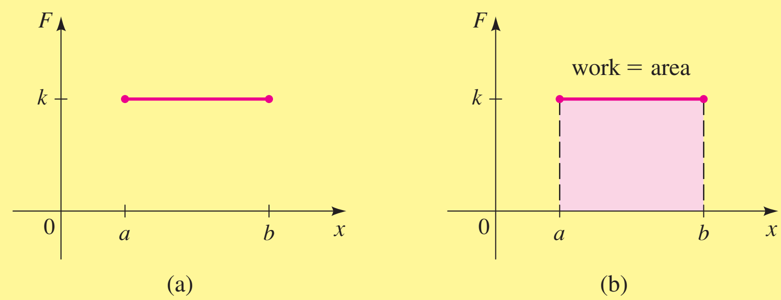


Figure 1
A constant force F

But what if the force is *not* constant? For example, suppose the force you apply to the crate varies with distance (you push harder at certain places than you do at others). More precisely, suppose that you push the crate along the x-axis in the positive direction, from $x = a$ to $x = b$, and at each point x between a and b you apply a force $f(x)$ to the crate. Figure 2 shows a graph of the force f as a function of the distance x.

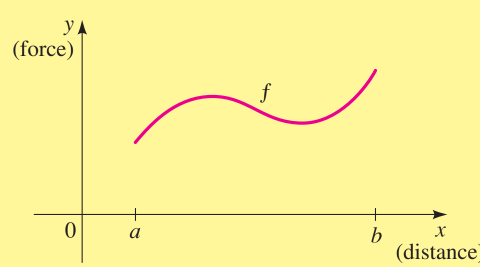


Figure 2
A variable force

How much work was done? We can't apply the formula for work directly because the force is not constant. So let's divide the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx as shown in Figure 3(a) on the next page. The force at the right endpoint of the interval $[x_{k-1}, x_k]$ is $f(x_k)$. If n is large, then Δx is small, so the values of f don't change very much over the interval $[x_{k-1}, x_k]$. In other words f is almost constant on the interval, and so the work W_k that is done in moving the crate from x_{k-1} to x_k is approximately

$$W_k \approx f(x_k)\,\Delta x$$

Thus, we can approximate the work done in moving the crate from $x = a$ to $x = b$ by

$$W \approx \sum_{k=1}^{n} f(x_k)\,\Delta x$$

It seems that this approximation becomes better as we make n larger (and so make the interval $[x_{k-1}, x_k]$ smaller). Therefore, we define the work done in moving an object from a to b as the limit of this quantity as $n \to \infty$:

$$W = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x$$

Notice that this is precisely the area under the graph of f between $x = a$ and $x = b$ as defined in Section 12.5. See Figure 3(b).

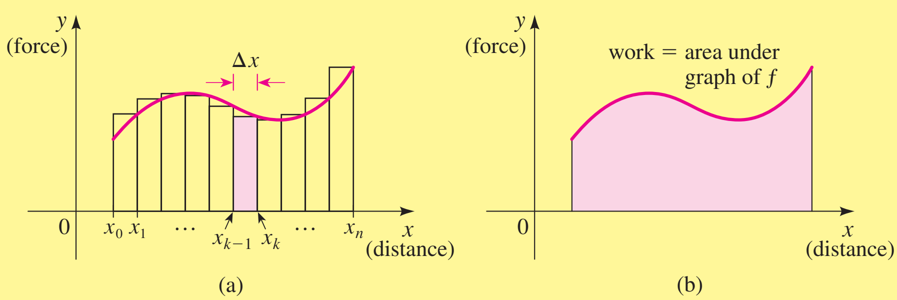


Figure 3
Approximating work

Example The Work Done by a Variable Force

A man pushes a crate along a straight path a distance of 18 ft. At a distance x from his starting point, he applies a force given by $f(x) = 340 - x^2$. Find the work done by the man.

Solution The graph of f between $x = 0$ and $x = 18$ is shown in Figure 4. Notice how the force the man applies varies—he starts by pushing with a force of 340 lb, but steadily applies less force. The work done is the area under the graph of f on

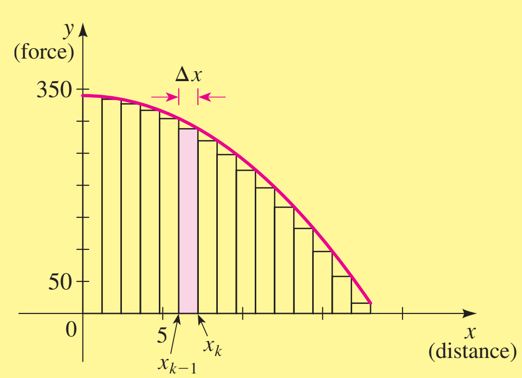


Figure 4

the interval $[0, 18]$. To find this area, we start by finding the dimensions of the approximating rectangles at the nth stage.

Width: $$\Delta x = \frac{b-a}{n} = \frac{18-0}{n} = \frac{18}{n}$$

Right endpoint: $$x_k = a + k\,\Delta x = 0 + k\left(\frac{18}{n}\right) = \frac{18k}{n}$$

Height: $$f(x_k) = f\left(\frac{18k}{n}\right) = 340 - \left(\frac{18k}{n}\right)^2 = 340 - \frac{324k^2}{n^2}$$

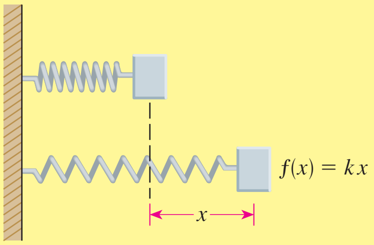
Thus, according to the definition of work we get

$$\begin{aligned} W &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\,\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left(340 - \frac{324k^2}{n^2}\right)\left(\frac{18}{n}\right) \\ &= \lim_{n\to\infty}\left(\frac{18}{n}\sum_{k=1}^{n} 340 - \frac{(18)(324)}{n^3}\sum_{k=1}^{n} k^2\right) \\ &= \lim_{n\to\infty}\left(\frac{18}{n}340n - \frac{5832}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]\right) \\ &= \lim_{n\to\infty}\left(6120 - 972\cdot\frac{n}{n}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}\right) \\ &= 6120 - 972\cdot 1\cdot 1\cdot 2 = 4176 \end{aligned}$$

So the work done by the man in moving the crate is 4176 ft-lb. ■

Problems

1. **Work Done by a Winch** A motorized winch is being used to pull a felled tree to a logging truck. The motor exerts a force of $f(x) = 1500 + 10x - \frac{1}{2}x^2$ lb on the tree at the instant when the tree has moved x ft. The tree must be moved a distance of 40 ft, from $x = 0$ to $x = 40$. How much work is done by the winch in moving the tree?

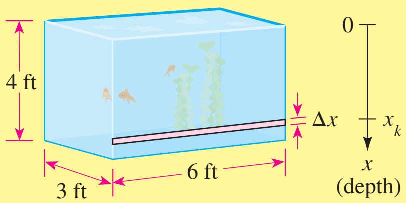


2. **Work Done by a Spring** Hooke's law states that when a spring is stretched, it pulls back with a force proportional to the amount of the stretch. The constant of proportionality is a characteristic of the spring known as the **spring constant**. Thus, a spring with spring constant k exerts a force $f(x) = kx$ when it is stretched a distance x.

   A certain spring has spring constant $k = 20$ lb/ft. Find the work done when the spring is pulled so that the amount by which it is stretched increases from $x = 0$ to $x = 2$ ft.

3. **Force of Water** As any diver knows, an object submerged in water experiences pressure, and as depth increases, so does the water pressure. At a depth of x ft, the water pressure is $p(x) = 62.5x$ lb/ft^2. To find the force exerted by the water on a surface, we multiply the pressure by the area of the surface:

$$\text{force} = \text{pressure} \times \text{area}$$



Suppose an aquarium that is 3 ft wide, 6 ft long, and 4 ft high is full of water. The bottom of the aquarium has area $3 \times 6 = 18$ ft^2, and it experiences water pressure of $p(4) = 62.5 \times 4 = 250$ lb/ft^2. Thus, the total force exerted by the water on the bottom is $250 \times 18 = 4500$ lb.

The water also exerts a force on the sides of the aquarium, but this is not as easy to calculate because the pressure increases from top to bottom. To calculate the force on one of the 4 ft by 6 ft sides, we divide its area into n thin horizontal strips of width Δx, as shown in the figure. The area of each strip is

$$\text{length} \times \text{width} = 6\ \Delta x$$

If the bottom of the kth strip is at the depth x_k, then it experiences water pressure of approximately $p(x_k) = 62.5x_k$ lb/ft^2—the thinner the strip, the more accurate the approximation. Thus, on each strip the water exerts a force of

$$\text{pressure} \times \text{area} = 62.5x_k \times 6\ \Delta x = 375x_k\ \Delta x \text{ lb}$$

(a) Explain why the total force exerted by the water on the 4 ft by 6 ft sides of the aquarium is

$$\lim_{n \to \infty} \sum_{k=1}^{n} 375x_k\ \Delta x$$

where $\Delta x = 4/n$ and $x_k = 4k/n$.

(b) What area does the limit in part (a) represent?

(c) Evaluate the limit in part (a) to find the force exerted by the water on one of the 4 ft by 6 ft sides of the aquarium.

(d) Use the same technique to find the force exerted by the water on one of the 4 ft by 3 ft sides of the aquarium.

NOTE Engineers use the technique outlined in this problem to find the total force exerted on a dam by the water in the reservoir behind the dam.

4. Distance Traveled by a Car Since distance = speed × time, it is easy to see that a car moving, say, at 70 mi/h for 5 h will travel a distance of 350 mi. But what if the speed varies, as it usually does in practice?

(a) Suppose the speed of a moving object at time t is $v(t)$. Explain why the distance traveled by the object between times $t = a$ and $t = b$ is the area under the graph of v between $t = a$ and $t = b$.

(b) The speed of a car t seconds after it starts moving is given by the function $v(t) = 6t + 0.1t^3$ ft/s. Find the distance traveled by the car from $t = 0$ to $t = 5$ s.

5. Heating Capacity If the outdoor temperature reaches a maximum of 90 °F one day and only 80 °F the next, then we would probably say that the first day was hotter than the second. Suppose, however, that on the first day the temperature was below 60 °F for most of the day, reaching the high only briefly, whereas on the second day the temperature stayed above 75 °F all the time. Now which day is the hotter one? To better measure how hot a particular day is, scientists use the concept of **heating degree-hour**. If the temperature is a constant D degrees for t hours, then the "heating capacity" generated over this period is Dt heating degree-hours:

$$\text{heating degree-hours} = \text{temperature} \times \text{time}$$

If the temperature is not constant, then the number of heating degree-hours equals the area under the graph of the temperature function over the time period in question.

(a) On a particular day, the temperature (in °F) was modeled by the function $D(t) = 61 + \frac{6}{5}t - \frac{1}{25}t^2$, where t was measured in hours since midnight. How many heating degree-hours were experienced on this day, from $t = 0$ to $t = 24$?

(b) What was the maximum temperature on the day described in part (a)?

(c) On another day, the temperature (in °F) was modeled by the function $E(t) = 50 + 5t - \frac{1}{4}t^2$. How many heating degree-hours were experienced on this day?

(d) What was the maximum temperature on the day described in part (c)?

(e) Which day was "hotter"?